# Algebraic error in the numerical solution of differential equations

*J. Liesen, J. Papež, Z. Strakoš*

Department of Mathematics, TU Berlin
Institute of Computer Science AS CR, Prague
Faculty of Mathematics and Physics, Charles University in Prague

## 1 Introduction

In the contribution we consider the discretization of differential equation by the finite element method (FEM) and using simple model problems we illustrate numerically that the algebraic error can create a significant local components and can dominate locally the total error, even when the globally measured algebraic error (in the energy norm or as the algebraic backward error) is significantly smaller than the globally measured discretization error.

The FEM generates an approximate solution of the model in form of a linear combination of functions with stricly *local* supports. The *global* approximation property of the FEM approximate solution is restored by solving a linear algebraic system for the coefficients of this linear combination. The fact that in practice we do not solve this system *exactly* can have fundamental consequences.

## 2 Model problem 2D

We consider the two-dimensional Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \equiv [0,1]^2\,, \quad f(x,y) = -2\,(x^2+y^2-x-y)\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

with the smooth solution

$$u(x,y) = x\,(x-1)\,y\,(y-1)$$

discretized using the Galerkin finite element method with linear basis functions on a regular triangular grid with 30 inner nodes in each direction.

We have computed the (approximate) solution of the system $Ax = b$ arised from the discretization using the MATLAB backslash operator. Neglecting the algebraic error in this computation, the (closely approximated) squared energy norm of the discretization error is

$$\|\nabla(u - u_h)\|^2 = 6.1588 \times 10^{-5}.$$

The shape of the discretization error is very similar to the shape of the solution, see Fig. 1.

We apply the conjugate gradient method (CG) to the linear algebraic system $Ax = b$ with initial guess $x_0 = 0$. By $u_h^{(n)}$ we denote the approximation to the Galerkin solution $u_h$ determined by the vector $x_n$ generated in the $n$-th iteration of the CG method. The errors satisfy the Galerkin orthogonality relation

$$\|\nabla(u - u_h^{(n)})\|^2 = \|\nabla(u - u_h)\|^2 + \|\nabla(u_h - u_h^{(n)})\|^2 = \|\nabla(u - u_h)\|^2 + \|x - x_n\|_A^2\,.$$

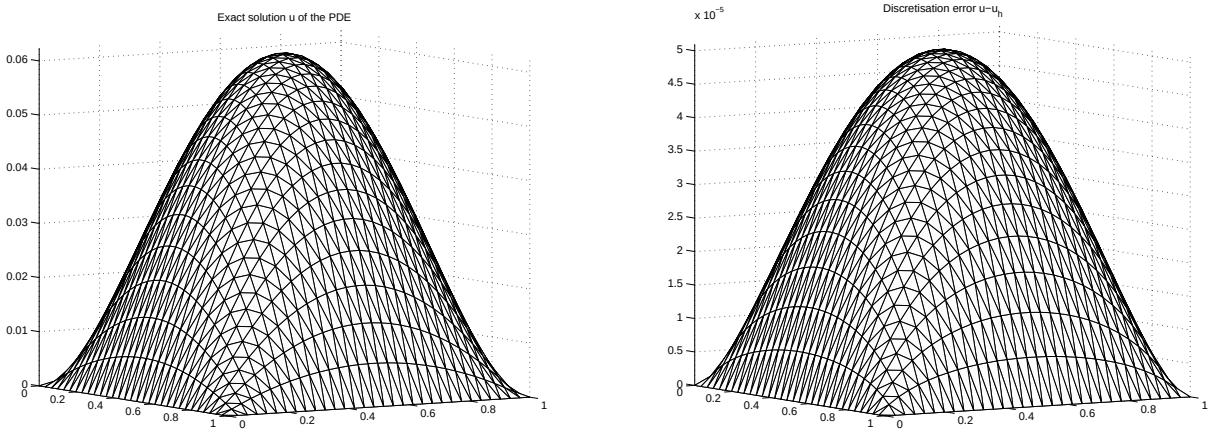


Figure 1: *Piecewise linear approximations to the solution of the model problem u (left) and to the discretization error* $u - u_h$ *(right).*

After 19 CG iteration steps the values of the total error and the algebraic error (measured in the energy norm or as the algebraic backward error) are

| Total error $\|\nabla(u - u_h^{(19)})\|^2$ | Algebraic error $\|x - x_{19}\|_A^2$ | Normwise backward error |
|---|---|---|
| $6.1774 \times 10^{-5}$ | $1.8586 \times 10^{-7}$ | $8.4672 \times 10^{-5}$ |

We can see that the total error measured in the energy norm is dominated by the discretization error.

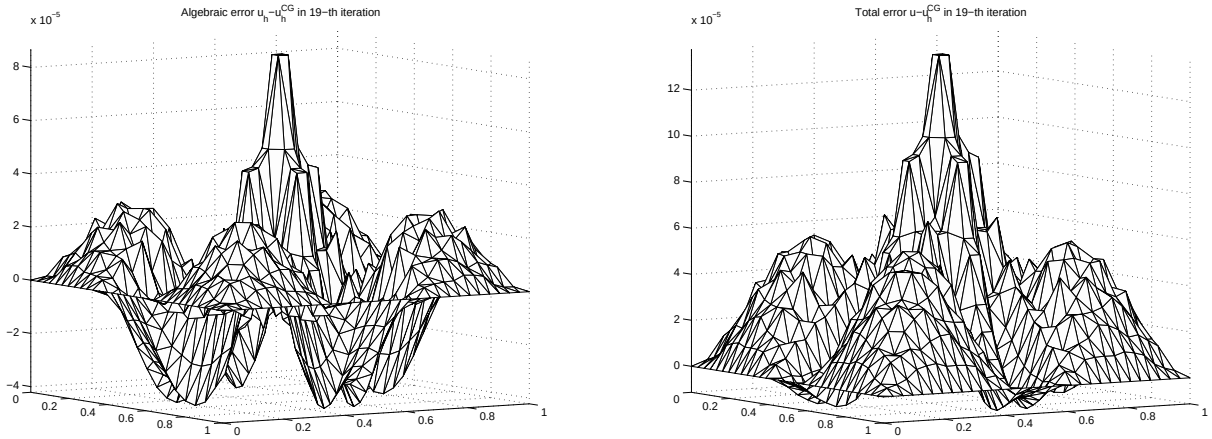


Figure 2: *Algebraic error (left) and piecewise linear approximation to the total error (right) in the model problem after 19 CG iteration steps; The vertical axis are scaled by* $10^{-5}$.

When one considers the local distribution of error, the whole picture changes. Figure 2 shows the algebraic and total errors after 19 CG iteration steps. Apparently, the local distribution of the discretization and the algebraic error is significantly different, and the local behaviour of the total error is dominated by the algebraic error.

Please note, that Figures 1, 2 are generated using the MATLAB `trisurf` command, which generates a *linear* triangular surface plot. The discretization and total errors are not as smooth

as suggested by the plots, but contains "bubbles" inside the triangles, which can be (depending on the size of the error) significant. In order to show the errors we therefore consider analogous problem in one dimension.

# 3 Model problem 1D

We consider the one-dimensional boundary value problem

$$\begin{aligned} -u''(x) &= -12x^2 + 12x + 2\,, \qquad x \in [0,1] \\ u(0) &= u(1) = 0 \end{aligned} \tag{1}$$

discretized using the piecewise linear basis functions on uniform partition with 19 inner nodes. Figure 3 shows the exact solution

$$u(x) = (x-2)\,(x-1)\,x\,(x+1)$$

and the corresponding discretization error $u - u_h$. The coeficients of $u_h$ are approximated (to a sufficient accuracy) using the MATLAB backslash operator. The values of $u - u_h$ at the nodes of the partition are on the machine precision level.

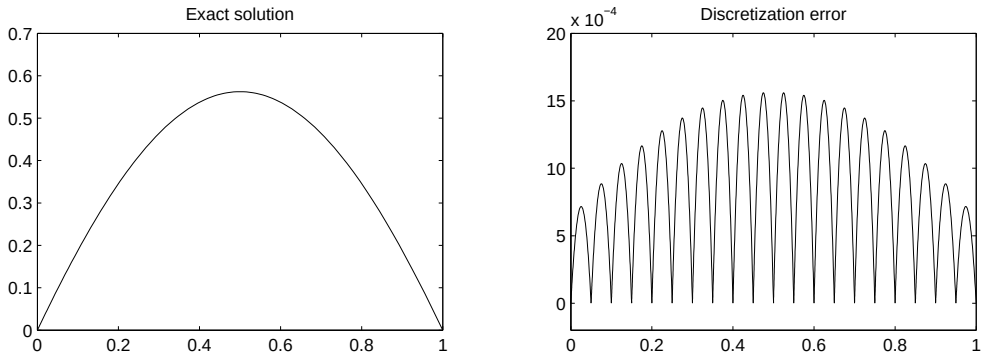


Figure 3: *The exact solution $u$ (left) and the discretization error $u - u_h$ (right) in the one-dimensional model problem.*

For solving the stiffness system we apply the CG method with $x_0 = 0$. After 9 CG iteration steps the values of the total error, the discretization error (measured in the energy norm) and the algebraic error (measured in the energy norm or as the algebraic backward error) are

| Total error $\|\nabla(u - u_h^{(9)})\|^2$ | Discretization error $\|\nabla(u_h - u_h^{(9)})\|^2$ | Algebraic error $\|x - x_9\|_A^2$ | Normwise backward error |
|---|---|---|---|
| $3.7556 \times 10^{-3}$ | $3.500 \times 10^{-3}$ | $2.5563 \times 10^{-4}$ | $8.5921 \times 10^{-4}$ |

The algebraic and total errors are shown in Fig. 4.

# 4 Conclusion

Let us first stress that we do not advocate to solve the Poisson problem on regular domains by the CG method. Our goal is to show on a simple model problem some important phenomena

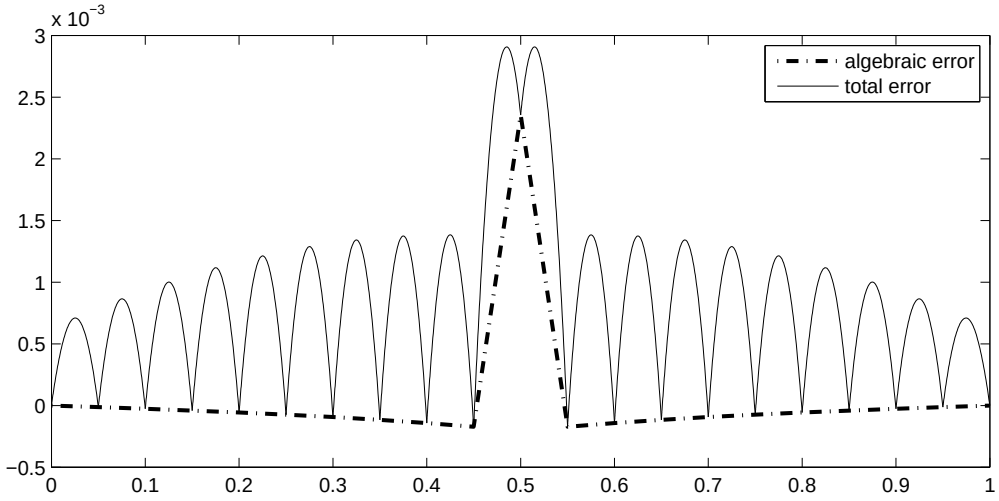


Figure 4: *The algebraic error (left) and the total error (right) in the one-dimensional model problem after 9 CG iteration steps.*

which should be taken into account when solving large scale mathematical modeling problems in general. It should be admitted that matrix computations can not be considered a separate (black box) part of the numerical solution process. Apart from relatively simple cases, black box approaches may not work. Even worse, they are philosophically wrong. Even if direct algebraic solvers are applicable, the resulting algebraic error might not be small and it should be considered (or the opposite should be rigorously justified). The stopping criteria in iterative algebraic solvers should be linked, in an optimal case, with fully computable and locally efficient (on individual elements) *a posteriori* error bounds that allow to keep an appropriate balance between the discretization and the algebraic parts of the error.

## References

[1] J. Liesen, Z. Strakoš: *Principles and Analysis of Krylov Subspace Methods*, Chapter 5; to be published by Oxford University Press.

[2] J. Papež: *Estimation of the algebraic error and stopping criteria in numerical solution of partial differential equations*. Master Thesis, Charles University in Prague, Faculty of Mathematics and Physics, 2011.