# Matching moments and matrix computations

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Abstract

Matching moments is inherently linked with numerical quadrature, continued fractions and orthogonal polynomials. Its relationship to matrix computations has a remarkable history. In this talk we will discuss how the classical topics of the 19th century mathematics found so widespread applications in solving modern computational problems formulated via matrices, and show that awareness of the underlined relationships can be beneficial in constructing and analysis of modern methods and algorithms.

The concept of moments arose with the work of Chebyshev, Markov and Stieltjes in the second half of the 19th century. The related numerical quadrature is even older. It started with Gauss (see the 1814 paper [1]), with further founding contributions due to Jacobi, Christoffel, Markov, Stieltjes and many others. The related fundamental concept of continued fractions can essentially be rooted back to Euclid and other ancient mathematicians. It is present in many mathematical disciplines such as number theory, approximation theory, probability and statistics, numerical quadrature, spectral theory, and modern matrix computations; see, e.g., [2]. The same is true for a slightly more recent concept of orthogonal polynomials. Among very many mathematicians who contributed in a fundamental way to the related developments one should not miss Brouncker and Wallis giving in 1655 what we now call the Stieltjes three-term recurrences, Euler with his overwhelming 18th century work including infinite series expansions of continued fractions, and Stieltjes with his analytic theory of continued fractions described in a truly fascinating paper [3] published in 1894. That paper also presented, as a by-product, a (Riemann-Stieljes) generalization of the Riemann integral, and a formal description of the moment problem together with its complete solution. We can observe an impact of the related results in forming foundations of functional analysis by Hilbert in 1906 - 1912, as well as in forming mathematical foundations of quantum mechanics by von Neumann in 1927 - 1932. Continued fractions and Jacobi matrices are still important in spectral theory of operators in mathematical physics and they are used in physics and computational chemistry.

The original work of Krylov from 1931 focused on computation of the minimal polynomial via a transformation of a secular equation [4], and it refers in a substantial way to the work of Jacobi [5] from 1846. Its algebraic formulation, with using what we now call the Krylov sequence, was given by Gantmacher in 1934 [6]. In modern computational mathematics, sciences and engineering, Krylov subspace methods and matching moments model reduction (in approximation of large scale dynamical systems and elsewhere) can be viewed as nothing but a translation of the classical concepts mentioned above to the language of large scale matrix computations. Surprisingly, several important works which made these links transparent remained almost unknown; see, in particular, the work of Vorobyev [7] and the description in [8]. More details are given in the review [9].

One can take the links mentioned above simply as a material worth of historical essays. We will, however, argue that there is more to learn here. Looking back has a positive impact on modern computations; examples can be found, e.g., in [10, 11]. Viewing relevant matrix computations as matching moments is inspirational and useful for understanding of behavior of methods and algorithms, and sometimes it can be useful for examining validity of approaches used in modern matrix computations. We will demonstrate this on several examples partially presented in the recent works [12, 13, 14]. In particular, we will address the questions of cost evaluation in Krylov

subspace iterations and of efficient numerical approximation of the bilinear form $c^* A^{-1} b$. Besides the analytic aspects we will show importance of numerical stability considerations.

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