The Golub-Kahan Iterative Bidiagonalization in Regularization of Ill-posed problems and Estimation of the Noise in the Data

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Abstract

A broad class of applications requires solving linear ill-posed approximation problems of the form

$$Ax \approx b$$

with a right-hand side b (observation vector) contaminated by noise. The matrix A often represents a discretized smoothing operator (such as a discretized blurring operator in image deblurring problems) with the singular values of A decaying gradually without a noticeable gap; A is usually ill-conditioned. The presence of the noise causes additional difficulties; the direct solution $A^{+}x$, where A^{+} denotes a matrix pseudoinverse, represents usually a meaningless *noise-dominated* solution. Therefore *regularization methods* are used for finding numerical approximations to the solution which reflect a sufficient amount of information contained in the data, while suppressing the devastating influence of the noise.

The Golub-Kahan iterative bidiagonalization belongs among popular techniques with regularization properties. Here the original problem is projected onto a lower dimensional (Krylov) subspace, which in fact represents a form of regularization by projection. The projected problem, however, inherits a part of the ill-posedness of the original problem, and therefore some inner regularization must be applied [5]. Stopping criteria for the whole process are usually based on the regularization of the projected (small) problem. Regularization parameters are typically determined by L-curve techniques, estimation of the distance between the exact and regularized solution, the discrepancy principle, cross validation methods (see, e.g., [1, Chap. 7, pp. 175–208] and [6] for comparison of these methods).

In this contribution we restrict ourselves to ill-posed problems where the right-hand side b is contaminated by *white noise*

$$b = b^{exact} + b^{noise}$$

with the unknown noise level; we only assume $\|b^{noise}\| \ll \|b^{exact}\|$. By the nature of the problem we can assume that multiplication of a vector v by A and A^T results in smoothing which reduces the relative sizes of the high frequency components in v. In addition we assume that (on average) the left singular vectors u_j of A represent increasing frequencies as j increases and the linear system satisfies the discrete Picard condition, i.e., the absolute value of the projections of the exact right-hand side b^{exact} to the left singular subspaces of A decays (on average) faster than the corresponding singular values. All given assumptions are natural for a broad class of ill-posed problems.

Based on the assumptions given above, it was shown in [4] how the noise contained in the right-hand side b is propagated to the projected problem in the Golub-Kahan iterative bidiagonalization. Similar ideas are used in [7, 8, 3] for selection of a value of the regularization parameter for which the residual vector changes from being dominated by the remaining signal to being white-noise like. This leads to a parameter-choice method based on Fourier analysis of residual vectors. The noise propagation to reconstructed images computed by regularizing iterations is studied in [2].

In [4], information about the noise propagation is further used for *estimating the unknown noise level* in the data from the information available during the bidiagonalization process. The noise level detection is connected with the Gauss quadrature approximation of the Rieman-Stieltjes

distribution function determined by the input data. The presented estimate is then based on monitoring the absolute value of the first component of the left singular vector of the bidiagonal matrix corresponding to its smallest singular value. It can be computed at a negligible cost. Its accuracy and robustness is investigated using various test problems.

After reviewing the results presented in [4], we turn into the problem of *approximating the unknown noise vector* b^{noise}. Such information could further be used for improving accuracy of a regularized solution and construction of efficient stopping criteria for the Golub-Kahan iterative bidiagonalization. The work is in progress.

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