DEDUCTIVE SYSTEMS OF FUZZY LOGIC

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(1) Fuzzy logic is popular.
(2) Logic studies the notion(s) of consequence.
(3) Fuzziness is impreciseness (vagueness); a fuzzy proposition may be true in some degree.
(4) The term "fuzzy logic" has two different meanings - wide and narrow.
(5) This lecture is about the fuzzy logic in the narrow sense.
(6) Where are the truth values from? First, why the unit interval [0,1]? And second, what does it mean that the truth degree of a proposition is 0.7?
(7) Most many-valued logics are truth-functional.
(8) Things to come.
- propositional calculi: basic, Łukasiewicz, Gödel, product; Pavelka extension of Łukasiewicz.
- predicate calculi: the same (more or less).

We hope to show:

- *Fuzzy logic is neither a poor man's logic nor poor man's probability.* Fuzzy logic (in the narrow sense) is a reasonably deep theory.
- *Fuzzy logic is a logic.* It has its syntax and semantic and notion of consequence. It is a study of consequence.
- *There are various systems of fuzzy logic, not just one.* The three basic systems are those of Łukasiewicz and of Gödel and the product logic.
- *Fuzzy logic in the narrow sense is a beautiful logic, but also is important for applications:* it offers foundations.

Fuzzy propositional calculi

Real truth values, t-norms and their residua.

Choices: (1) *propositional logic.*
(2) $[0, 1]$ the set of truth values, 1 being absolute truth, 0 absolute falsity.

(3) *Truth-functionality:* Choosing truth functions of connectives: each many-valued logic must be a generalization of classical two-valued logic.
(4) we start by requirements on a truth function for *conjuction*. We shall see that a proper choice of the semantics of conjunction determines a whole propositional calculus.

Big truth degree of $\varphi \& \psi$ – indicates that both the truth degree of φ and the truth degree of ψ is big.

Definition. A *t-norm* is a binary operation $*$ on $[0,1]$ satisfying the following conditions:

(i) t is commutative and associative, i.e., for all $x, y, z \in [0,1]$,

$$\begin{aligned} x * y &= y * x, \\ (x * y) * z &= x * (y * z), \end{aligned}$$

(ii) t is non-decreasing in both arguments, i.e.

$$\begin{aligned} x_1 \leq x_2 &\quad \text{implies} \quad x_1 * y \leq x_2 * y, \\ y_1 \leq y_2 &\quad \text{implies} \quad x * y_1 \leq x * y_2, \end{aligned}$$

(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.

$*$ is a *continuous t-norm* if it is a t-norm and is a continuous mapping of $[0,1]^2$ into $[0,1]$ (in the usual sense).

Example

The following are our most important examples of continuous t-norms:

(i) *Lukasiewicz t-norm*:
$x * y = \max(0, x + y - 1)$,

(ii) *Gödel t-norm:*
$x * y = \min(x, y)$,

(iii) *Product t-norm:*
$x * y = x.y$ (product of reals).

Implication. A big truth-value of $\varphi \rightarrow \psi$ – indicates that the truth value of φ is *not too bigger* than the truth value of ψ.

Fuzzy modus ponens: from (a lower bound of) the truth degree x of φ and (a lower bound of) the truth degree $x \Rightarrow y$ of $(\varphi \rightarrow \psi)$ one should be able to compute a lower bound of the truth degree y of ψ.

$$x * z \leq y \text{ IFF } z \leq x \Rightarrow y).$$

Then $x \Rightarrow y$ is the *maximal* z satisfying
$x * z \leq z$.

The operation $\Rightarrow$ is called the *residuum* of the t-norm $*$.

Properties

(i) $x \leq y$ iff $(x \Rightarrow y) = 1$

(ii) $(1 \Rightarrow x) = x$

Residua of the above t-norms:

$x \Rightarrow y = 1$ for $x \leq y$ and, for $x < y$

(i) *Lukasiewicz implication*:
$x \Rightarrow y = 1 - x + y$

(ii) *Gödel implication*:
$x \Rightarrow y = y$

(iii) *Goguen implication*:
$x \Rightarrow y = y/x$

for $x > y$.

Negation: Lukasiewicz $(-x) = 1 - x$

Gödel and Goguen: $(-)0 = 1$, $(-)x = 0$ for $x > 0$

$\cap$ and $\cup$ denote min and max respectively. Note that the ordering $\leq$ is definable: $x \leq y$ iff $x \cap y = x$. We show that min and max are definable from $*$ and $\Rightarrow$.

For each continuous t-norm $*$, the following identities are true in $L(*)$:

(i) $x \cap y = x * (x \Rightarrow y)$,

(ii) $x \cup y = ((x \rightarrow y) \rightarrow y) \cap ((y \rightarrow x) \rightarrow x)$.

x is idempotent if $x * x = x$.

$*$ continuous t-norm: the set E of all its idempotents is a closed subset of [0,1] and hence its complement is a union of a set $\mathcal{I}(E)$ of countably many non-overlapping open intervals.

Lemma. For each $I \in \mathcal{I}(E)$, $*|I$ is isomorphic either to the product t-norm (on [0,1]) or to Lukasiewicz t-norm (on [0,1]). If $x, y \in [0,1]$ are such that there is no $I \in \mathcal{I}(E)$ with $x, y \in I$ then $x * y = \min(x, y)$.

$x *_{CP} y = \max(\frac{1}{4}, x.y)$ (product cut at $\frac{1}{4}$).

Lemma. The ordered semigroup $\langle [\frac{1}{4}, 1], *_{CP} \rangle$ is isomorphic to $\langle [0,1], *_{\mathrm{L}} \rangle$ where $*_{\mathrm{L}}$ is Lukasie- wicz t-norm $x *_{\mathrm{L}} y = \max(0, x + y - 1)$.

The basic fuzzy logic

Fix a continuous t-norm $*$: you fix a propositional calculus (whose set of truth values is $[0,1]$): $*$ is the truth function of the (strong) conjunction &, the residuum $\Rightarrow$ of $*$ becomes the truth function of the implication. Further connectives are defined as follows:

$$\begin{aligned}
\varphi \wedge \psi &\quad \text{is} \quad \varphi \& (\varphi \rightarrow \psi), \\
\varphi \vee \psi &\quad \text{is} \quad ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi), \\
\neg \varphi &\quad \text{is} \quad \varphi \rightarrow \bar{0}, \\
\varphi \equiv \psi &\quad \text{is} \quad (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi).
\end{aligned}$$

An *evaluation of propositional variables* is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0,1]$.

This extends:

$$e(\bar{0}) = 0,$$
$$e(\varphi \to \psi) = (e(\varphi) \Rightarrow e(\psi)),$$
$$e(\varphi \& \psi) = (e(\varphi) * e(\psi)).$$

A formula φ is a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e.

The following formulas are axioms of the basic logic:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \& \psi) \to \varphi$
(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$
(A4) $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to$
$(((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\bar{0} \to \varphi$

The *deduction rule* of BL is modus ponens. Given this, the notions of a *proof* and of a *provable formula* in BL are defined in the obvious way

All axioms of BL are 1-tautologies in each $PC(*)$. If φ and $\varphi \to \psi$ are 1-tautologies of $PC(*)$ then ψ is also a 1-tautology of $PC(*)$. Consequently, each formula provable in BL is a 1-tautology of each $PC(*)$.

BL proves the following properties of implication:

(1) $\varphi \to (\psi \to \varphi)$
(2) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(3) $\varphi \to \varphi$

BL proves the following properties of strong conjunction:

(4) $(\varphi \& (\varphi \to \psi)) \to \psi$
(5) $\varphi \to (\psi \to (\varphi \& \psi))$
(6) $(\varphi \to \psi) \to ((\varphi \& \chi) \to (\psi \& \chi))$
(7) $((\varphi_1 \to \psi_1) \& (\varphi_2 \to \psi_2)) \to$
$((\varphi_1 \& \varphi_2) \to (\psi_1 \& \psi_2))$
(8) $(\varphi \& \psi) \& \chi \to \varphi \& (\psi \& \chi)$.

BL proves the following properties of min-conjunction:

(9) $(\varphi \wedge \psi) \to \varphi, (\varphi \wedge \psi) \to \psi, (\varphi \& \psi) \to (\varphi \wedge \psi)$
(10) $(\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$
(11) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$
(12) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

BL proves the following properties of max-disjunction:

(13) $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$, $(\varphi \vee \psi) \to (\psi \vee \varphi)$,
(14) $(\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$
(15) $(\varphi \to \psi) \vee (\psi \to \varphi)$

(16) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to (\varphi \vee \psi) \to \chi$.

BL proves

(11') $((\varphi \to \psi)\&(\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$,

(16') $(\varphi \to \chi)\&(\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$.

BL proves the following properties of negation.

(17) $\varphi \to (\neg\varphi \to \psi)$, in particular, $\varphi \to \neg\neg\varphi$

(18) $(\varphi \to (\psi\&\neg\psi)) \to \neg\varphi$

(18') $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

$\bar{1}$ stands for $\bar{0} \to \bar{0}$.

BL proves the following:

(19) $\bar{1}$,

(20) $\varphi \to (\bar{1}\&\varphi)$.

BL proves the following additional properties of $\wedge, \vee$:

(21) $(\varphi \wedge (\psi \wedge \chi)) \to ((\varphi \wedge \psi) \wedge \chi)$
$((\varphi \wedge \psi) \wedge \chi) \to (\varphi \wedge (\psi \wedge \chi))$
(associativity of $\wedge$),

(22) analogous associativity for $\vee$,

(23) $\varphi \to \varphi \wedge (\varphi \vee \psi)$
$(\varphi \vee (\varphi \wedge \psi)) \to \varphi$

BL proves

(24) $\varphi \equiv \varphi, \quad (\varphi \equiv \psi) \to (\psi \equiv \varphi)$,
$((\varphi \equiv \psi)\&(\psi \equiv \chi)) \to (\varphi \equiv \chi)$,

(25) $(\varphi \equiv \psi) \to (\varphi \to \psi)$,
$(\varphi \equiv \psi) \to (\psi \to \varphi)$

(26) $(\varphi \equiv \psi) \to ((\varphi\&\chi) \equiv (\psi\&\chi))$,

(27) $(\varphi \equiv \psi) \to ((\varphi \to \chi) \equiv (\psi \to \chi))$,

(28) $(\varphi \equiv \psi) \to ((\chi \to \varphi) \equiv (\chi \to \psi))$,

(29) $(\varphi \equiv \psi) \to ((\varphi \to \psi) \wedge (\psi \to \varphi))$.

A *theory* over BL is a set of formulas. A *proof* in a theory T is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas whose each member is either an axiom of BL or a member of T (special axiom) or follows from some preceding members of the sequence using the deduction rule modus ponens.

$T \vdash \varphi$ means that φ is *provable* in T, i.e. is the last member of of a proof in T.

Deduction theorem: Let T be a theory and let φ, ψ be formulas.

$T \cup \{\varphi\} \vdash \psi$ iff there is an n such that $T \vdash \varphi^n \to \psi$ (where φ^n is $\varphi\& \ldots \&\varphi$, n factors).

BL proves the following distributive laws:

(30) $\varphi\&(\psi \vee \chi) \equiv (\varphi\&\psi) \vee (\varphi\&\chi)$
$\varphi\&(\psi \wedge \chi) \equiv (\varphi\&\psi) \wedge (\varphi\&\chi)$

(31) $(\varphi \wedge (\psi \vee \chi)) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
$(\varphi \vee (\psi \wedge \chi)) \equiv ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$

BL proves:

(32) $(\varphi \vee \psi)\&(\varphi \vee \psi) \to ((\varphi\&\varphi) \vee (\psi\&\psi))$
$(\varphi \wedge \psi)\&(\varphi \wedge \psi) \to ((\varphi\&\varphi) \wedge (\psi\&\psi))$

(33) $(\varphi \to \psi)^n \vee (\psi \to \varphi)^n$, for each n,
where α^n is $\alpha\& \ldots \alpha$, n times.

BL-agebras; a completeness theorem.

A *BL-algebra* is an algebra

$$(L, \cap, \cup, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

(i) $(L, \cap, \cup, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative, $1 * x = x$ for all x.

(iii) the following conditions hold:

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z.
(2) $x \cap y = x * (x \Rightarrow y)$
(3) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

The class of all BL-algebras is a primitive class of algebras (a variety)

An **L**-evaluation of propositional variables: any mapping e assigning to each propositional variable p an element $e(p)$ of **L**. This extends in the obvious may to an evaluation of all formulas using the operations on **L** as truth functions.

The logic BL is sound with respect to **L**-tautologies: if φ is provable in BL then φ is an **L**-tautology for each BL-algebra. More generally, if T is a theory over BL and T proves φ then, for each BL-algebra **L** and each **L**-evaluation e of propositional variables assigning the value 1 to all the axioms of T, $e(\varphi) = 1$.

Classes of provably equivalent formulas (w.r.t. a theory T) form a BL-algebra BL-algebra.

Filters on BL-algebras determine homomorphisms and prime filters characterize homomorphisms to linearly ordered BL-algebras.

A *filter* on **L** is a set $F \subseteq L$ such that for each $x, y \in L$,
$a \in F$ and $b \in F$ implies $a * b \in F$,
$a \in F$ and $a \leq b$ implies $b \in F$.

F is an *prime filter* iff for each $x, y \in L$,
$(x \Rightarrow y) \in F$ or $(y \Rightarrow x) \in F$.

Let **L** be a BL-algebra BL-algebra and let $a \in L, a \neq 1$. Then there is a prime filter F on **L** not containing a.

Each BL-algebra BL-algebra is a subalgebra of the direct product of a system of linearly ordered regular BL-algebras.

BL is complete, i.e. for each formula φ the following there things are equivalent:

(i) φ is provable in BL,

(ii) for each linearly ordered BL-algebra BL-algebra $\mathbf{L}$, φ is an $\mathbf{L}$-tau-tology;

(iii) for each BL-algebra BL-algebra $\mathbf{L}$, φ is an $\mathbf{L}$- tautology.

We shall generalize this completeness theorem as follows:

(1) An *axiom schema* given by a formula
$\Phi(p_1, \ldots, p_n)$ is the set of all formulas
$\Phi(\varphi_1, \ldots, \varphi_n)$ resulting by the substitution of φ_i for $p_i (i = 1, \ldots, n)$ in $\Phi(p_1, \ldots, p_n)$.

(2) A logical calculus $\mathcal{C}$ is a *schematic extension* of BL if it results from BL by adding some (finitely or infinitely many) axiom schemata to its axioms. (The deduction rule remains to be modus ponens.)

(3) Let $\mathcal{C}$ be a schematic extension of BL and let $\mathbf{L}$ be a BL-algebra. $\mathbf{L}$ is a *$\mathcal{C}$-algebra* if all axioms of $\mathcal{C}$ are $\mathbf{L}$-tautologies.

Completeness. Let $\mathcal{C}$ be a schematic extension of BL and let φ be a formula. The following are equivalent:

(i) $\mathcal{C}$ proves φ,

(ii) φ is an $\mathbf{L}$-tautology for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$,

(iii) φ is an $\mathbf{L}$-tautology for each $\mathcal{C}$-algebra $\mathbf{L}$.

Łukasiewicz propositional logic

BL + $(\neg\neg)$, where

$(\neg\neg)$ $\neg\neg\varphi \to \varphi$. Lukasiewicz's axioms:

(L1) $\varphi \to (\psi \to \varphi)$,

(L2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(L3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

(L4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

This axiom system will be denoted by Ł.

Lemma BL + $(\neg\neg)$ proves (Ł1) to (Ł4).

Conversely, Lukasiewicz's axioms prove axioms of BL plus $(\neg\neg)$.

$$\neg\varphi \text{ is } \varphi \to \bar{0},$$
$$\varphi\&\psi \text{ is } \neg(\varphi \to \neg\psi).$$

New connectives $\wedge, \vee$ and $\underline{\vee}$ are introduced in Ł as follows:

$$\begin{array}{rll} \varphi \wedge \psi & \text{stands for} & \varphi\&(\varphi \to \psi), \\ \varphi \vee \psi & \text{stands for} & (\varphi \to \psi) \to \psi, \\ \varphi \underline{\vee} \psi & \text{stands for} & \neg\varphi \to \psi. \end{array}$$

Note that (Ł4) just expresses the commutativity of $\vee$.

Ł proves the following properties of $\wedge$, $\vee$, $\underline{\vee}$ (note that (8)–(11) are de Morgan rules):

(8) $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
(9) $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
(10) $\neg(\varphi \& \psi) \leftrightarrow (\neg\varphi \underline{\vee} \neg\psi)$
(11) $\neg(\varphi \underline{\vee} \psi) \leftrightarrow (\neg\varphi \& \neg\psi)$
(12) $\psi \to (\varphi \underline{\vee} \psi)$
(13) $(\varphi \underline{\vee} \psi) \to (\psi \underline{\vee} \varphi)$
(14) $(\varphi \underline{\vee} (\psi \underline{\vee} \chi))$
$\leftrightarrow ((\varphi \underline{\vee} \psi) \underline{\vee} \chi)$
(15) $(\varphi \wedge \psi) \leftrightarrow (\varphi \underline{\vee} \neg\psi) \& \psi$
(16) $(\varphi \vee \psi) \leftrightarrow (\varphi \& \neg\psi) \underline{\vee} \psi$
(17) $\varphi \underline{\vee} \neg\varphi$
(18) $((\varphi \& \neg\psi) \underline{\vee} \psi) \leftrightarrow (\varphi \underline{\vee} (\psi \& \neg\varphi))$
(19) $((\varphi \underline{\vee} \neg\psi) \& \psi) \leftrightarrow (\varphi \& (\psi \underline{\vee} \neg\varphi))$
(20) $(\varphi \vee \varphi) \to \varphi$
(21) $(\varphi \to \chi) \to (\psi \to \chi) \to ((\varphi \vee \psi) \to \chi)$

Lemma Ł proves $(\varphi \to \psi) \vee (\psi \to \varphi)$.

MV-algebras; a completeness theorem

An *MV-algebra* is a BL-algebra BL-algebra is which the identity

$$((x \Rightarrow y) \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow x) = 1$$

is valid (MV stands for "many-valued").

Let $\mathbf{G} = \langle G, \oplus, \ominus, 0_G, \leq_G \rangle$ be a linearly ordered Abelian group (loag) and let $e \in G$, $0 <_G e$ be a positive element. $MA(\mathbf{G}, e)$ is the algebra $\mathbf{M} = \langle M, \Rightarrow, 0_G \rangle$ whose domain M is the interval $[0, e]_G = \langle g \in G \mid 0 \leq_G g \leq e \rangle$, $x \Rightarrow y = e$ if $x \leq_G y$ and $x \Rightarrow y = e \ominus x \oplus y$ otherwise.

Lemma. $MA(\mathbf{G}, e)$ is a linearly ordered MV-algebra.

Conversely, for each linearly ordered MV-algebra $\mathbf{M}$ there is a linearly ordered Abelian froup $\mathbf{G}$ and its positive element e such that $\mathbf{M} = MV(G, e)$.

Lemma. If an identity $\sigma = \tau$ in the language of MV-algebras is valid in the standard MV-algebra $[0, 1]$ with truth functions then it is valid in each linearly ordered MV-algebra. (Consequenly, if a formula φ is a 1-tautology over the standard MV-algebra then φ is an $\mathbf{M}$-tautology for each linearly ordered MV-algebra.

Theorem. Let $[0, 1]_{Ł}$ denote the standard MV-algebra on $[0, 1]$ with truth functions of Łukasiewicz logic.

(1) A formula φ is a 1-tautology of Łukasie- wicz logic Ł (i.e. a $[0, 1]_{Ł}$-tautology) iff it is an $\mathbf{M}$-tautology for each linearly ordered MV-algebra $\mathbf{M}$.

(2) Let T be a finite theory over Ł. The following are equivalent.

(2i) φ is true in each $[0, 1]_{Ł}$-model of T, (2ii) for each linearly ordered MV-algebra

$\mathbf{M}$, φ is true in each $\mathbf{M}$-model of T.

Standard completeness.

(1) A formula φ is provable in Łukasiewicz logic Ł iff it is a 1-tautology of Łukasiewicz logic.

(2) Let T be a finite theory over Ł, φ a formula. T proves φ over Łukasiewicz logic iff φ is true in each model of T (in the sense of Ł).

It is easy to show that we cannot omit the assumption of finiteness of T.

Rational Pavelka logic

One may ask: can we also prove *partially* true conclusions from *partially* true premisses? Introduce, for each *rational* $r \in [0,1]$, a *truth constant* $\bar{r}$ — a special formula whose truth value under each evaluation is r. Thus we shall have $e(\psi) \geq r$ iff $e(\bar{r} \rightarrow \psi) = 1$; $e(\psi) \leq r$ iff $e(\psi \rightarrow \bar{r}) = 1$.

The *axioms* of RPL are the axioms of Ł (thus (Ł1)–(Ł4)) plus the following bookkeeping axioms for truth constants:

$$\begin{aligned} (\bar{r} \rightarrow \bar{s}) &\equiv \overline{r \Rightarrow s} \\ \neg\bar{r} &\equiv \overline{1-r} \end{aligned}$$

(e.g. $\overline{0.8} \rightarrow \overline{0.7} = \overline{0.9}$, $\neg\overline{0.4} = \overline{0.6}$). The deduction rule is modus ponens.

A *theory* is a set of formulas — special axioms. Proofs and provability are as usual. A *graded formula* is a pair (φ, r) where φ is a formula and r a rational element of $[0,1]$; it is just another notation for the formula $(\bar{r} \rightarrow \varphi)$.

The deduction theorem remains valid for RPL: $T \cup \{\varphi\} \vdash \psi$ iff for some n $T \vdash \varphi^n \rightarrow \psi$, φ^n being $\varphi \& \ldots \& \varphi$.

Let T be a theory over RPL, φ a formula.

(1) The *truth degree* of φ over T is $\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a model of } T\}$.

(2) The *provability degree* of φ over T is $|\varphi| = \sup\{r \mid T \vdash (\varphi, r)\}$.

(Completeness.) For each theory T and each formula φ, the provability degree equals the truth degree, i.e.

$$|\varphi|_T = \|\varphi\|_T.$$

(1) A theory T is *consistent* if for each $r < 1$ does not prove $\bar{r}$. (In other words, for each $s > 0$, the formula $\bar{s} \rightarrow \bar{0}$ is improvable, $|\bar{0}|_T = 0$.)

(2) A theory T is *complete* if for each formula φ and each r, $T \vdash \varphi \rightarrow \bar{r}$ or $T \vdash \bar{r} \rightarrow \varphi$.

T is inconsistent iff $T \vdash \varphi$ for each φ.

Lemma. For each consistent theory T there is a consistent complete supertheory $T' \supseteq T$.

Lemma. Let T be consistent and complete. Then the provability degree commutes with connectives, i.e.

$$|\neg\varphi|_T = 1 - |\varphi|_T, \ \ |\varphi \to \psi|_T = |\varphi|_T \Rightarrow |\psi|_T.$$

Thus the evaluation $e(p_i) = |p_i|_T$ is a model of T.

We show that if desired we can get rid of truth constants, replacing them by some special axioms over Ł. (In fact, it is more elegant to keep the constants; the reduction is useful for some future proofs.)

Associate with each rational $r \in [0,1]$ a propositional variable p_r. For each r there is a formula φ_r of Ł (i.e. not containing truth constants) built from p_r and atmost two other propositional variables p_s, p_t such that for each evaluation e, $e(\varphi_r) = 1$ iff $e(p_r) = r$.

For example, let φ_0 be $p_0 \equiv p_0 \& \neg p_0$, φ_1 be $p_1 \equiv p_1 \underline{\vee} \neg p_1$ and $\varphi_{\frac{1}{2}}$ be $p_{\frac{1}{2}} \equiv \neg p_{\frac{1}{2}}$.

Let φ^* result from φ by replacing any truth constant $\bar{r}$ by the variable p_r and let $\hat{T}$ be the theory in Ł whose axioms are all the axioms φ_r Let $q_1, \ldots$ be variables distinct from all p_r. For each theory T in the q-variables, let $T^{**} = T^* \cup \hat{T}$ where $T^* = \{\alpha^* | \alpha \in T\}$. For each q-formula φ, $T \vdash_{RPL} \varphi$ iff $T^{**} \vdash_{Ł} \varphi^*$; thus theories in RPL reduce to theories in Ł.

We shall use our (cumbersome) analysis to prove the following strong completeness theorem for RPL:

Theorem. Let T be a finite theory over RPL let φ be a formula. If φ is true in all models of T then $T \vdash \varphi$.

We show that RPL may be expanded by the product conjunction.

RPL($\odot$) stands for RPL whose language results from that of RPL by adding a new binary connective $\odot$, interpreted as the product conjunction: $x.y$. The axioms are extended by the following:

$$(\varphi \to \psi) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$$
$$(\varphi \to \psi) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$$
$$\bar{r} \odot \bar{s} \equiv \overline{r.s} \text{ (for all rational } r, s \in [0,1]).$$

Completeness. For each theory T over RPL($\odot$), $|\varphi|_T = \|\varphi\|_T$.

Continuity was crucial; e.g. one cannot have this style of completeness when adding the non-continuous implication of product logic.

Product logic.

We write $\odot$ instead of &.

The *axioms* of Π are those of BL plus

(Π1) $\neg\neg\chi \to ((\varphi \odot \chi \to \psi \odot \chi) \to (\varphi \to \psi))$,

(Π2) $\varphi \wedge \neg\varphi \to \bar{0}$.

The axioms are 1-tautologies over the algebra $[0,1]_P$ of the truth functions.

Π proves the following formulas:
(1) $\neg(\varphi \odot \psi) \to \neg(\varphi \wedge \psi)$
(2) $(\varphi \to \neg\varphi) \to \neg\varphi$
(3) $\neg\varphi \vee \neg\neg\varphi$

The axiom (Π2) can be equivalently replaced by each of the following formulas:

$$\neg(\varphi \odot \varphi) \to \neg\varphi, \ (\varphi \to \neg\varphi) \to \neg\varphi, \ \neg\varphi \vee \neg\neg\varphi.$$

Following the general approach we define a Π-*algebra* (or *product algebra*) to be a BL-algebra BL-algebra satisfying

$$\begin{gathered} \neg\neg z \leq ((x * z \Rightarrow y * z) \Rightarrow (x \Rightarrow y)), \\ x \cap -x = 0. \end{gathered}$$

Trivially, the class of all product algebras is a variety and Π is sound with respect to product algebras, i.e. each formula provable on Π is a **L**-tautology for each product algebra **L**.

Let $\mathbf{L} = \langle L, *, \Rightarrow, \cap, \cup, 0_L, 1_L \rangle$ be a linearly ordered product algebra. Then there is a linearly ordered Abelian group $\mathbf{G} = \langle G, \oplus_G, 0_G, \leq_G \rangle$ such that $L - \{0\} = Neg_G = \{g \in G | g \leq_G 0_G\}$ such that, for all $g, h \in L - \{0\}$,

$$\begin{gathered} 0_G = 1_L \\ g \oplus_G h = g * h \\ g \leq_G h \text{ iff } g \leq h \end{gathered}$$

Furthemore, for $g \geq h$, $g \Rightarrow h = h \ominus_G g$.

If an identity $\tau = \sigma$ in the language of product algebras is valid in the standard product algebra $[0,1]_P$ with truth functions then it is valid in all linearly ordered product algebras.

Completeness.
(1) A formula φ is provable in the product logic Π iff it is a 1-tautology of the product logic.
(2) Let T be a finite theory over Π, φ a formula. T proves φ over the product logic iff it is true in each model of T (in the sense of Π).

Gödel logic

The axiom system of G is the extension of the axiom system of BL by the single axiom

(G) $\qquad \varphi \to (\varphi \& \varphi)$

stating (together with $(\varphi \& \varphi) \to \varphi$) the idempotence of &.

G proves $(\varphi \& \psi) \equiv (\varphi \wedge \psi)$.

Thus we may get rid in G of one conjunction, say &; thus we may present Gödel logic equivalently as follows:

The axiom system G' has the connectives $\wedge$, $\rightarrow$, 0 and axioms (A1–A7) of BL, with & replaced by $\wedge$ plus the axiom (G'$_\Pi$): $\varphi \rightarrow (\varphi \wedge \varphi)$. Other connectives are defined as follows:

$$\neg\varphi \text{ is } \varphi \rightarrow \bar{0},\ (\varphi \equiv \psi) \text{ is } (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi),$$
$$(\varphi \vee \psi) \text{ is } ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi).$$

The *intuitionist logic* I has the connectives $\rightarrow, \wedge, \vee, \neg$ and the following axioms:

(I1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(I2) $\varphi \rightarrow (\varphi \vee \psi)$
(I3) $\psi \rightarrow (\varphi \vee \psi)$
(I4) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)))$
(I5) $(\varphi \wedge \psi) \rightarrow \varphi$
(I6) $(\varphi \wedge \psi) \rightarrow \psi$
(I7) $(\chi \rightarrow \varphi) \rightarrow ((\chi \rightarrow \psi) \rightarrow (\chi \rightarrow (\varphi \wedge \psi)))$
(I8) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \wedge \psi) \rightarrow \chi)$
(I9) $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(I10) $(\varphi \wedge \neg\varphi) \rightarrow \psi$
(I11) $(\varphi \rightarrow (\psi \wedge \neg\psi)) \rightarrow \neg\psi$

Theorem. G proves all axioms of I.

G has the classical deduction theorem: for each theory T over G and formulas φ, ψ,
$T \cup \{\varphi\} \vdash \psi$ iff $T \vdash (\varphi \rightarrow \psi)$.

BL-algebras satisfying the identity
$x * x = x$ (i.e. with idempotent multiplication) are called *G-algebras.*

If H is linearly ordered G-algebra then each subset of H containing 0_H and 1_H is a subalgebra.

Standard completeness:
(1) A formula φ of Gödel logic G is provable in G iff it is a 1-tautology of G.
(2) For each theory T over G and each formula φ, T proves φ (over G) iff φ is true in each model of T (over G).

We shall now discuss the behaviour of Gödel logic with respect to partial truth.

Theorem. For each finite theory T over G, each formula φ and each rational r such that $0 < r \leq 1$, $T \vdash \varphi$ iff each evaluation e such that $e(\alpha) \geq r$ for each axiom $\alpha \in T$ satisfies $e(\varphi) \geq r$ (i.e. if e makes all axioms r-true then it makes φ r-true).

Fuzzy predicate calculi.

The basic fuzzy predicate logic

A *predicate language*: *predicates* $P, Q, \ldots$, each together with its *arity* and *object constants.* $c, d, \ldots$. *Logical symbols* are *object variables* $x, y, \ldots$, *connectives* $\&, \rightarrow$, *truth constants* $\bar{0}, \bar{1}$ and *quantifiers* $\forall, \exists$. Other connectives ($\wedge$, $\vee$, $\neg$, $\equiv$) are defined as in prop. calculus. *Terms* are object variables and object constants.

Atomic formulas have the form $P(t_1, \ldots, t_n)$ where P is a predicate of arity n and $t_1, \ldots, t_n$ are terms. If φ, ψ are formulas and x is an object variable then $\varphi \rightarrow \psi$, $\varphi \& \psi$, $(\forall x)\psi$, $(\exists x)\varphi$, $\bar{0}$, $\bar{1}$ are formulas; each formula results from atomic formulas by iterated use of this rule.

Let $\mathcal{J}$ be a predicate language and let $\mathbf{L}$ be a BL-algebra BL-algebra. An $\mathbf{L}$-*structure* $\mathbf{M} = \langle M, (r_P)_P, (m_c)_c \rangle$ *for* $\mathcal{J}$, $M \neq \emptyset$, for each n-ary predicate P a $\mathbf{L}$-fuzzy n-ary relation $r_P : M^n \rightarrow \mathbf{L}$ on M and for each object constant c, m_c is an element of M.

An $\mathbf{M}$-*evaluation* of *object variables...*
$||x||_{M,v} = v(x)$; $||c||_{M,v} = m_c$.

$||P(t_1, \ldots, t_n)||^L_{M,v} = r_p(||t_1||_{M,v}, \ldots, ||t_n||_{M,v})$;
$||\varphi \rightarrow \psi||^L_{M,v} = ||\varphi||^L_{M,v} \Rightarrow ||\psi||^L_{M,v}$;
$||\varphi \& \psi||^L_{M,v} = ||\varphi||^L_{M,v} * ||\psi||^L_{M,v}$;
$||\bar{0}||_{M,v}$; $\qquad ||\bar{1}||_{M,v}$;
$||(\forall x)\varphi||^L_{M,v} = \inf\{||\varphi||^L_{M,v'} | v \equiv_x v'\}$;
$||(\exists x)\varphi||^L_{M,v} = \sup\{||\varphi||^L_{M,v'} | v \equiv_x v'\}$;
provided the infimum/supremum exists in the sense of $\mathbf{L}$;

The structure $\mathbf{M}$ is $\mathbf{L}$-*safe* if all the needed infima and suprema exist, i.e. $||\varphi||^{\mathbf{L}}_{M,v}$ is defined for all φ, v.

$||\varphi||^{bfL}_M = \inf\{||\varphi||^{\mathbf{L}}_{M,v} | \, v \;\; \mathbf{M} - \text{evaluation}\}$.

A formula φ of a language $\mathcal{J}$ is an $\mathbf{L}$-*tautology* if $||\varphi||^{\mathbf{L}}_M = 1_{\mathbf{L}}$ for each safe $\mathbf{L}$-structure $\mathbf{M}$. —bf M is an $\mathbf{L}$-*model* of a theory T if

The following are *logical axioms on quantifiers:*
($\forall$1) $(\forall x)\varphi(x) \rightarrow \varphi(t)$ (t substitutable for x in $\varphi(x)$)
($\exists$1) $\varphi(t) \rightarrow (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
($\forall$2) $(\forall x)(\nu \rightarrow \varphi) \rightarrow (\nu \rightarrow (\forall x)\varphi)$ (x not free in ν)
($\exists$2) $(\forall x)(\varphi \rightarrow \nu) \rightarrow ((\exists x)\varphi \rightarrow \nu)$ (x not free in ν)
($\forall$3) $(\forall x)(\varphi \vee \nu) \rightarrow ((\forall x)\varphi \vee \nu)$ (x not free in ν)

Predicate calculus $\mathcal{C}\forall$ (over a given predicate language $\mathcal{J}$):

- all formulas resulting from the axioms of $\mathcal{C}$ by substituting arbitrary formulas of $\mathcal{J}$ for propositional variables, and
- the axioms $(\forall 1), (\forall 2), (\exists 1), (\exists 2), (\forall 3)$ for quantifiers

  and deduction rules
- modus ponens (from φ, $\varphi \to \psi$ infer ψ) and
- generalization (from φ infer $(\forall x)\varphi$).

In particular, BL$\forall$ and three stronger logics: Ł$\forall$ (Łukasiewicz), G$\forall$ (Gödel), $\Pi\forall$ (product).

The axioms $(\forall 1)$–$(\forall 3)$, $(\exists 1)$ – $(\exists 2)$ are **L**-tautologies for each BL-algebra BL-algebra **L**.

(Soundness of provability.) Let $\mathcal{C}$ be a schematic extension of BL, let T be a theory in the language of T over $\mathcal{C}\forall$, let φ be a formula of T. If $T \vdash \varphi$ (φ is provable in T) then $||\varphi||^L_M = 1$ for each $\mathcal{C}$-algebra **L** and each **L**-model **M** of T.

Let φ be an arbitrary formula, ν a formula not containing x freely. Then BL$\forall$ proves the following:

(1) $(\forall x)(\nu \to \varphi) \equiv (\nu \to (\forall x)\varphi)$
(2) $(\forall x)(\varphi \to \nu) \equiv ((\exists x)\varphi \to \nu)$
(3) $(\exists x)(\nu \to \varphi) \to (\nu \to (\exists x)\varphi)$
(4) $(\exists x)(\varphi \to \nu) \to ((\forall x)\varphi \to \nu)$

The converse implications in (3), (4) are *not* provable in BL. Neither of them is a tautology of G$\forall$; the converse of (3) is but the converse of (4) is not a tautology of $\Pi\forall$; and both converses are tautologies of Ł$\forall$.

For arbitrary formulas φ, ψ, BL$\forall$ proves the following:

(5) $(\forall x)(\varphi \to \psi) \to ((\forall x)\varphi \to (\forall x)\psi)$
(6) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to (\exists x)\psi)$
(7) $((\forall x)\varphi \& (\exists x)\psi) \to (\exists x)(\varphi \& \psi)$

For arbitrary φ and for ν not containing x freely, BL$\forall$ proves
(9) $(\exists x)(\varphi \& \nu) \equiv ((\exists x)\varphi \& \nu)$,
(10) $(\exists x)(\varphi \& \varphi) \equiv ((\exists x)\varphi \& (\exists x)\varphi)$.

BL$\forall$ proves the following:
(11) $(\exists x)\varphi \to \neg(\forall x)\neg\varphi$
(12) $\neg(\exists x)\varphi \to (\forall x)\neg\varphi$

Completeness

Let T be a theory over $\mathcal{C}\forall$.

(1) T is *consistent* if there is a formula φ unprovable in T.

(2) T is *complete* if for each pair φ, ψ of closed formulas, $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$.

(3) T is *Henkin* if for each closed formula of the form $(\forall x)\varphi(x)$ unprovable in T there is a constant c in the language of T such that $\varphi(c)$ is unprovable in T.

For each theory T over $\mathcal{C}\forall$ we let $\mathbf{L}_T$ to be the algebra of classes of T-equivalent closed formulas with the usual operations ($[\varphi]_T \Rightarrow [\psi]_T = [\varphi \to \psi]_T$ etc.). Clearly, $\mathbf{L}_T$ is a $\mathcal{C}$-algebra.

(1) If T is complete then $\mathbf{L}_T$ is linearly ordered.

(2) If T is Henkin then for each formula $\varphi(x)$ with just one free variable x,

$$[(\forall x)\varphi]_T = \inf_c[\varphi(c)]$$
$$[(\exists x)\varphi]_T = \sup_c[\varphi(c)]$$

(c running over all constants of T).

*Lemma.*For each theory T and each closed formula α, if $T \not\vdash \alpha$ then there is a complete Henkin supertheory $\hat{T}$ of T such that $\hat{T} \not\vdash \alpha$.

Lemma. For each complete Henkin theory T and each closed formula φ unprovable in T there is a linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and an $\mathbf{L}$-model $\mathbf{M}$ of T such that $\|\varphi\|_M < 1_{\mathbf{L}}$.

Completeness. Let $\mathcal{C}\forall$ be the predicate calculus given by a schematic extension $\mathcal{C}$ of BL, let T be a theory over $\mathcal{C}\forall$ let φ be a formula of the language of T. T proves φ iff for each linearly ordered $\mathcal{C}$-algebra $\mathbf{L}$ and each $\mathbf{L}$-model of T, $\|\varphi\|_M^L = 1_{\mathbf{L}}$.

Łukasiewicz predicate logic

Ł$\forall$ proves

$$(\exists x\varphi) \equiv \neg(\forall x)\neg\varphi.$$

$$(\forall x)(\varphi \& \nu) \to ((\forall x)\varphi \& \nu).$$

Lemma. Ł$\forall$ proves the following:

$$(\nu \to (\exists x)\varphi) \to (\exists x)(\nu \to \varphi),$$
$$((\forall x)\varphi \to \nu) \to (\exists)(\varphi \to \nu).$$

Theorem. There is no recursive axiomatic system standardly complete with respect to Ł$\forall$-tautologies (over $[0,1]_{\text{Ł}}$).

Predicate rational Pavelka logic RPL∀.

Formulas: from predicates, variables, object constants and rational truth constants using connectives and quantifier(s).
Atomic formulas: as in Ł∀ plus truth constants.
Models: as in Ł∀. Satisfaction: as in Ł∀ plus

$$\| \overline{r} \|_{M,v} = r.$$

Axioms: those of RPL plus $(\forall 1), (\forall 2)$.
Rules: modus ponens and generalization. Provability and satisfiability degree: as in RPL.

Pavelka style completeness: Provability degree equals to satisfiability degree, i.e. for each theory over RPL∀

$$\| \varphi \|_T = |\varphi|_T.$$

Proof by transforming the canonical model of an RPL∀-theory T (over the MV-algebra of classes of T-equivalent formulas) to a model over $[0,1]_{Ł}$ putting, for each φ,

$$f(\varphi) = sup\{r | T \vdash (\overline{r} \to \varphi).$$

Remark: Continuity of the L-implication is crucial for the proof.

Caution: A formula φ is a 1-tautology of RPL∀ iff for each rational $r < 1$, $T \vdash (\overline{r} \to \varphi)$. But there are 1-tautologies φ such that $T \not\vdash \varphi$. This is essential.

Predicate product logic.

It is known that there is no recursive axiomatization standardly complete for predicate product logic. No analogon of RPL∀ for the predicate product logic is known (and no *direct* analogon is possible due to non-continuity of the implication of the product logic.)

Predicate Gödel logic.

The system of axioms of $G\forall$ is standardly complete for $G\forall$, i.e. a formula is provable in $G\forall$ iff it is a $[0,1]_G$-tautology.