GENERALIZED GRAM–SCHMIDT-BASED APPROXIMATE INVERSE PRECONDITIONING FOR THE CONJUGATE GRADIENT METHOD

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Abstract

This contribution deals with an approximate inverse preconditioning for the conjugate gradient method. In particular, it focuses on the generalized Gram–Schmidt process. Its algorithm is performed incompletely which means that some computed entries (small in some sense) are dropped. Assume a system of linear equations in the form $Ax = b$, where A is symmetric and positive definite. Symmetrically preconditioned system can be written in the form

$$\tilde{Z}^T A\tilde{Z}y = \tilde{Z}^T b, \qquad x = \tilde{Z}y,$$

where $\tilde{Z}$ is the factor of the approximation $\tilde{Z}\tilde{Z}^T$ to A^{-1}, that plays the role of the preconditioner. It seems that the A-orthogonality of the column vectors of the matrix $\tilde{Z}$ measured by the norm $\|\tilde{Z}^T A\tilde{Z} - I\|$ (the loss of A-orthogonality among column vectors of the matrix $\tilde{Z}$) and sparsity of the preconditioner reflected in the number of nonzeros of $\tilde{Z}$ indicate usefulness of the preconditioner.

Exact version of the generalized Gram–Schmidt process provides matrices Z and U, so that $U^T U = (Z^{(0)})^T AZ^{(0)}$, $Z^T AZ = I$, and $ZU = Z^{(0)}$. Columns of the matrix $Z^{(0)}$ are initial vectors that are A-orthogonalized against previously computed vectors. Matrix U is composed from the orthogonalization coefficients. It is clear, that for $Z^{(0)} = I$ the matrix U is equal to the Cholesky factor of the matrix $A = U^T U$. The bounds for the norms $\|\bar{Z}^T A\bar{Z} - I\|$, $\|\bar{Z}\bar{U} - I\|$, $\|\bar{U}^T\bar{U} - (Z^{(0)})^T AZ^{(0)}\|$ for the main orthogonalization schemes, where the quantities with an extra bar are computed in the finite precision arithmetic, can be found in [1]. As for dropping, original schemes for the generalized Gram–Schmidt process have been introduced in [2]. These dropping rules considered magnitudes of matrix entries absolutely or with respect to some intermediate quantities. The rules were successful in practice but they lack theoretical justification. Note that up to now, theory of incomplete decompositions supports mainly their rather special cases.

Construction of an incomplete decomposition supported by theoretical background is the subject of this contribution. The analysis in [1] motivates development of new rules to drop entries in incomplete generalized Gram-Schmidt algorithm such that the computed factors have similar properties as obtained from the standard finite precision algorithm. In our case, the role of the roundoff unit is played by the drop

tolerance. In order to improve numerical properties of the computational schemes, we introduce additional pivotal strategies and demonstrate their usefulness. We hope that the resulting algorithms may extend scope of applicability of the considered type of approximate inverse preconditioning.

References

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