Ratios of estimate versus exact condition

We now assume we have both \( R \) and \( R^{-1} \). Then we can for instance run ICE on \( R^{-1} \) and use the additional estimations

\[
\frac{1}{\sigma_i(R^{-1})} = \sigma_i(R) \quad \text{and} \quad \frac{1}{\sigma_i(R)} = \sigma_i(R^{-1})
\]

Surprisingly, this gives no better incremental estimator (see (B), Theorem 3.2):  

**Theorem 1**  
Let \( R \) be a nonsingular upper triangular matrix. Then the ICE estimates of the singular values of \( R \) and \( R^{-1} \) satisfy

\[
\frac{1}{\sigma_i(R^{-1})} = \lambda_i(R) \quad \text{and} \quad \frac{1}{\sigma_i(R)} = \lambda_i(R^{-1})
\]

The approximate left singular vectors \( y_i \) and \( \hat{y}_i \) correspond to the ICE estimates for \( R \) and \( R^{-1} \), respectively, satisfying

\[
\frac{1}{\sigma_i(R^{-1})} \approx \frac{1}{\sigma_i(R)}
\]

One can prove \( \sigma_i(R^{-1}) = o(\sigma_i(R)) \) as well. Using the inverse does improve ICE estimation (\( \sigma \)), Theorem 3.2).

**Theorem 2**  
Let \( R \) be a nonsingular upper triangular matrix. Assume that the ICE estimates of the singular values of \( R \) and \( R^{-1} \) are exact

\[
\frac{1}{\sigma_i(R^{-1})} = \lambda_i(R) = \sigma_i(R)
\]

Then the ICE estimates of the singular values related to the ice incremental estimate satisfy

\[
\frac{1}{\sigma_i(R^{-1})} \approx \frac{1}{\sigma_i(R)}
\]

with equality if and only if \( x \) is collinear with the left singular vector for the smallest singular value of \( R \).

In case the assumption is relaxed to \( \frac{1}{\sigma_i(R^{-1})} \leq \sigma_i(R) \) we obtain a rather technical theorem, saying essentially that maximization with \( R^{-1} \) is in most cases superior to minimization with \( R \). Nevertheless, in all performed numerical experiments we found that \( \sigma_i(R^{-1}) \) gives an estimate which is worse than \( \sigma(R) \).

**Figure 1** : ICE estimation of the smallest singular value for the ID Laplacian of size one until hundred: ICE with minimization (solid line), INE with maximization exploiting the inverse (circles) and exact minimum singular values (crosses). Hence if the inverse of \( R \) is available, it is recommendable to use \( 1/\sigma_i(R^{-1}) \) instead of \( \sigma_i(R) \) to approximate the minimum singular value. An analogue of Theorem 2 for estimates of the maximum singular value shows that

\[
\frac{1}{\sigma_i(R^{-1})} \leq \frac{1}{\sigma_i(R)}
\]

Thus for the maximum singular value, it is recommendable to use the original \( \sigma_i(R) \) instead of \( 1/\sigma_i(R) \). In this sense, INE performs better when doing maximization than when doing minimization.

**Figure 2** : Zoom of Figure 1. ICE estimation of the smallest singular value for the ID Laplacian of size fifty until hundred for ICE with maximization exploiting the inverse (circles) and exact maximum singular values (crosses).

### Numerical experiments

We will compare the following estimators:

- The original ICE technique with the estimates defined as \( \sigma_i(R^{-1})/\sigma_i(R) \). Solid lines.
- The original ICE technique with the estimates defined by \( \sigma_i(R^{-1})/\sigma_i(R) \). Circles.
- The ICE technique based on maximization only, i.e. estimates defined as \( \sigma_i(R^{-1})/\sigma_i(R) \). Plusses.
- The ICE technique based on minimization only, i.e. estimates defined as \( \sigma_i(R)/\sigma_i(R^{-1}) \). Squares.

**Example 1** : The same experiments as in [1, Section 4, Test 2], [2, Section 5, Table 5.4].

**Example 2** : 20 moderate size matrices from the Matrix Market collection, most of them tested also in [1, Section 5, Table 5.1]. We computed their QR decomposition (with and without column pivoting) and tested the estimators with the factor \( R \).

**Figure 3** : Ratios of estimate versus exact condition number for Example 1 with \( c = 100 \).

**Figure 4** : Ratios of estimate versus exact condition number for Example 1 with \( c = 1000 \).

**Figure 5** : Ratios of estimate versus exact condition number for Example 2 using column pivoting.

**Figure 6** : Ratios of estimate versus exact condition number for Example 2 without column pivoting.

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### Future work

We wish to investigate in particular more in detail the following issues:

- Large sparse matrices: It may be possible to obtain the same accurate estimates without storing the (in general dense) inverse triangular factors. But computation of the inverse factors seems to be unavoidable.
- Block versions based on block factorizations for dense matrices to enable exploitation of fast BLAS techniques.
- Improved incremental condition estimation by the Incremental estimation method on the Matrix Market database, [10, pp. 250-252].

**References**


