On prescribing Ritz values and GMRES residual norms generated by Arnoldi processes

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joint work with

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3. Prescribing convergence behavior for *both* Arnoldi's method and GMRES

1. Motivation

Given a nonsingular matrix and nonzero vector

$$A \in \mathbb{C}^{n \times n}, \quad b \in \mathbb{C}^n,$$

the kth iteration of the Arnoldi orthogonalization process [Arnoldi - 1951] (without breakdown) computes the decomposition

$$AV_k = V_{k+1}\tilde{H}_k,$$

where the columns of $V_k = [v_1, \ldots, v_k]$ (the Arnoldi vectors) contain an orthogonal basis for the kth Krylov subspace

$$\mathcal{K}_k(A, b) \equiv \text{span}\{b, Ab, \ldots, A^{k-1}b\}$$

and $\tilde{H}_k \in \mathbb{C}^{(k+1)\times k}$ is rectangular upper Hessenberg; by deleting its last row we get the square matrix

$$H_k = V_k^* A V_k \in \mathbb{C}^{k \times k}.$$

1. Motivation

Essentially,

- for eigenpair approximations of A, the Arnoldi method [Arnoldi - 1951], [Saad - 1980] uses the eigenvalues (so-called Ritz values) and eigenvectors of H_k and the first k Arnoldi vectors,
- for approximate solutions to linear systems $Ax = b$, the GMRES method [Saad, Schultz - 1986] solves least squares problems with $\tilde{H}_k$ and $||b||e_1$ and the first k Arnoldi vectors.

- Both the GMRES and the Arnoldi method are very popular methods that are successful for a large variety of problem classes.
- Nevertheless, convergence behavior of the two methods is not fully understood, analysis is particularly challenging with highly non-normal input matrices.

1. Motivation

- Often one tries to use the tools that are successful for analysis of hermitian counterparts of GMRES and Arnoldi like the Conjugate Gradient and the Lanczos method.
- For example, the basic tool for explaining Krylov subspace methods for hermitian linear systems is the eigenvalue distribution.
- However, for GMRES it is known for some time that if GMRES generates a certain residual norm history, the same history can be generated with any nonzero spectrum [Greenbaum, Strakoš - 1994].
- Complemented with the fact that GMRES can generate arbitrary non-increasing residual norms, this gives the result that any non-increasing convergence curve is possible with any nonzero spectrum [Greenbaum , Pták, Strakoš - 1996].
- A complete description of the class of matrices and right hand sides with prescribed convergence and eigenvalues was given in [Arioli , Pták, Strakoš - 1998].

1. Motivation

An important tool for hermitian eigenproblems solved with Krylov subspace methods is the following interlacing property:

Consider a tridiagonal Jacobi matrix T_m and its leading principal submatrix T_k for some $k < m$. If the ordered eigenvalues of T_k are

$$\rho_1^{(k)} < \rho_2^{(k)} < \quad \dots \quad < \rho_k^{(k)},$$

then in every open interval between two subsequent eigenvalues

$$(\rho_{i-1}^{(k)}, \rho_i^{(k)}), \quad i = 2, \dots, k,$$

there lies at least one eigenvalue of T_m.

This interlacing property enables, among others, to prove the persistence theorem (see [Paige - 1971, 1976, 1980] or [Meurant, Strakoš - 2006]) which is crucial for controlling the convergence of Ritz values in the Lanczos method.

1. Motivation

- There are generalizations of the interlacing property to the non-hermitian but normal case [Fan, Pall - 1957], [Thompson- 1966], [Ericsson - 1990], [Malamud - 2005], though a geometric interpretation is difficult.
- There is no interlacing property for the eigenvalues of principal submatrices of general non-normal matrices [de Oliveira - 1969], [Shomron, Parlett - 2009].
- This makes convergence analysis of the Arnoldi method for non-normal input matrices rather delicate, just as it is for the GMRES method.
- The GMRES and Arnoldi methods being closely related through the Arnoldi process, can we show that arbitrary convergence behavior of Arnoldi is possible?
- By arbitrary behavior we mean arbitrary Ritz values for *all* iterations (we do not consider eigenvectors). Note that this involves many more conditions than prescribing one residual norm per GMRES iteration.

2. Prescribed convergence for Arnoldi's method

Notation: Let the kth Hessenberg matrix H_k generated in Arnoldi's method have the eigenvalue ρ and normalized eigenvector y,

$$H_k\, y = \rho\, y.$$

With the Arnoldi decomposition $AV_k = V_{k+1}\tilde{H}_k$, the Ritz pair

$$\{\rho, V_k y\},$$

i.e. the Ritz value - Ritz vector pair has residual norm

$$\|A(V_k y) - \rho(V_k y)\| = \|A(V_k y) - V_k H_k y\| = \|V_{k+1}\tilde{H}_k y - V_k H_k y\| = h_{k+1,k}|e_k^T y|.$$

Often for small $h_{k+1,k}|e_k^T y|$, the Arnoldi method takes $\{\rho, V_k y\}$ as an approximate eigenvalue-eigenvector pair of A. Note that a small value $h_{k+1,k}|e_k^T y|$ needs not imply that ρ is close to a true eigenvalue of A, see e.g. [Chatelin - 1993], [Godet-Thobie - 1993]; convergence analysis cannot be based on this value but focusses instead on the quality of approximate invariant subspaces [Beattie, Embree, Sorensen - 2005].

2. Prescribed convergence for Arnoldi's method

Theorem 1 [DT, Meurant - 2012]. Let the set

$$\mathcal{R} = \{ \quad \rho_1^{(1)}, \\ (\rho_1^{(2)}, \rho_2^{(2)}), \\ \vdots \\ (\rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}), \\ (\lambda_1, \ldots\ldots\ldots, \lambda_n) \},$$

represent any choice of $n(n+1)/2$ complex Ritz values and denote by $C^{(k)}$ the companion matrix of the polynomial with roots $\rho_1^{(k)}, \ldots, \rho_k^{(k)}$, i.e.

$$C^{(k)} = \begin{pmatrix} 0 & \ldots & & 0 & -\alpha_0 \\ 1 & 0 & \ldots & 0 & -\alpha_1 \\ & \ddots & & \vdots & \vdots \\ & & & 1 & -\alpha_{k-1} \end{pmatrix}, \qquad \prod_{j=1}^{k}(z - \rho_j^{(k)}) = z^k + \sum_{j=0}^{k-1} \alpha_j z^j.$$

2. Prescribed convergence for Arnoldi's method

If we define the unit upper triangular matrix $U(\mathcal{S})$ through

$$U(\mathcal{S}) = I_n - \begin{bmatrix} 0 & C^{(1)}e_1 & \vdots & \vdots \\ & 0 & C^{(2)}e_2 & \vdots \\ & & 0 & \vdots \\ & & & C^{(n-1)}e_{n-1} \\ & & & 0 \end{bmatrix},$$

then the upper Hessenberg matrix

$$H(\mathcal{R}) = U(\mathcal{S})^{-1} C^{(n)} U(\mathcal{S})$$

has the spectrum $\lambda_1, \ldots, \lambda_n$ and its kth leading principal submatrix has spectrum

$$\rho_1^{(k)}, \ldots, \rho_k^{(k)}, \qquad k = 1, \ldots, n-1.$$

It has unit subdiagonal.

2. Prescribed convergence for Arnoldi's method

Proof: The $k \times k$ leading principal submatrix of $H(\mathcal{R})$ is

$$[I_k, 0]\, H(\mathcal{R}) \begin{bmatrix} I_k \\ 0 \end{bmatrix} = [I_k, 0]\, U(\mathcal{S})^{-1} C^{(n)} U(\mathcal{S}) \begin{bmatrix} I_k \\ 0 \end{bmatrix}$$

$$= \quad [U_k^{-1}, \tilde{u}_{k+1}, \ldots, \tilde{u}_n] \begin{bmatrix} 0 \\ U_k \\ 0 \end{bmatrix} = [U_k^{-1}, \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ U_k \end{bmatrix},$$

where U_k denotes the $k \times k$ leading principal submatrix of $U(\mathcal{S})$ and $\tilde{u}_j$ denotes the vector of the first k entries of the jth column of $U(\mathcal{S})^{-1}$ for $j > k$. Its spectrum is also the spectrum of the matrix

$$U_k [U_k^{-1}, \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ U_k \end{bmatrix} U_k^{-1} = [I_k, U_k \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ I_k \end{bmatrix},$$

which is a companion matrix with last column $U_k \tilde{u}_{k+1}$.

2. Prescribed convergence for Arnoldi's method

From

$$e_{k+1} = U_{k+1}U_{k+1}^{-1}e_{k+1} = \begin{bmatrix} U_k & -C^{(k)}e_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_{k+1} \\ 1 \end{bmatrix} = \begin{bmatrix} U_k\tilde{u}_{k+1} - C^{(k)}e_k \\ 1 \end{bmatrix}$$

we obtain $U_k\tilde{u}_{k+1} = C^{(k)}e_k$. $\square$

Remark: The matrix

$$H(\mathcal{R}) = U(\mathcal{S})^{-1}C^{(n)}U(\mathcal{S}).$$

is the *unique* upper Hessenberg matrix $H(\mathcal{R})$ with the prescribed spectrum and Ritz values and the entry one along the subdiagonal (see also [Parlett, Strang - 2008] where $H(\mathcal{R})$ is constructed in a different way).

Note that $U(\mathcal{S})$ transforms the matrix $C^{(n)}$ with all Ritz values zero to the matrix $H(\mathcal{R})$ with prescribed Ritz values. It is composed of (columns of) companion matrices and we will call $U(\mathcal{S})$ the Ritz value companion transform.

2. Prescribed convergence for Arnoldi's method

Thus the Ritz values generated in the Arnoldi method can exhibit any convergence behavior: Apply the Arnoldi method to the pair

$$\{H(\mathcal{R}), e_1\}$$

where $H(\mathcal{R})$ has the desired Ritz values. Then the method generates the Hessenberg matrix $H(\mathcal{R})$ itself.

Question: Can the same prescribed Ritz values be generated with positive entries other than one on the subdiagonal?

For $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} > 0$ consider the diagonal similarity transformation

$$H \equiv \text{diag}\left(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j\right) H(\mathcal{R}) \left(\text{diag}\left(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j\right)\right)^{-1}.$$

Then the subdiagonal of H is $\sigma_1, \ldots, \sigma_{n-1}$ and all leading principal submatrices of H are similar to corresponding leading principal submatrices of $H(\mathcal{R})$.

2. Prescribed convergence for Arnoldi's method

This immediately leads to a parametrization of the matrices and initial Arnoldi vectors that generate a given set of Ritz values $\mathcal{R}$:

Theorem 2 [DT, Meurant - 2012]. Assume we are given a set of tuples

$$\begin{aligned}\mathcal{R} = \{ \quad & \rho_1^{(1)}, \\ & (\rho_1^{(2)}, \rho_2^{(2)}), \\ & \vdots \\ & (\rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}), \\ & (\lambda_1, \ldots\ldots\ldots, \lambda_n)\}, \end{aligned}$$

of complex numbers and $n-1$ positive real numbers

$$\sigma_1, \ldots, \sigma_{n-1}.$$

If A is a matrix of order n and b a nonzero n-dimensional vector, then the following assertions are equivalent:

2. Prescribed convergence for Arnoldi's method

1. The Hessenberg matrix generated by the Arnoldi method applied to A and initial Arnoldi vector b has eigenvalues $\lambda_1, \ldots, \lambda_n$, subdiagonal entries $\sigma_1, \ldots, \sigma_{n-1}$ and $\rho_1^{(k)}, \ldots, \rho_k^{(k)}$ are the eigenvalues of its kth leading principal submatrix for all $k = 1, \ldots, n-1$.
2. The matrix A and initial vector b are of the form

$$A = V D_\sigma U(\mathcal{S})^{-1} C^{(n)} U(\mathcal{S}) D_\sigma^{-1} V^*, \qquad b = \|b\| V e_1,$$

where V is unitary, $U(\mathcal{S})$ is the Ritz value companion transform,

$$D_\sigma = \text{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j),$$

and $C^{(n)}$ is the companion matrix of the polynomial with roots $\lambda_1, \ldots, \lambda_n$.

This also shows how little on the quality of the Ritz value ρ needs be said by

$$\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k} |e_k^T y|.$$

Any distance from ρ to the spectrum of A is possible with any value of $h_{k+1,k}$!

2. Prescribed convergence for Arnoldi's method

Counterintuitive example 1: Convergence of interior Ritz values only:

$$\begin{aligned}\mathcal{R} = \{ \quad & 3, \\ & (3,3)\,, \\ & (2,3,4)\,, \\ & (3,3,3,3)\,, \\ & (1,2,3,4,5)\}\,.\end{aligned}$$

This gives the unit upper Hessenberg matrix

$$H(\mathcal{R}) = U(\mathcal{S})^{-1} C^{(5)} U(\mathcal{S}) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ & 1 & 3 & -1 & 0 \\ & & 1 & 3 & 5 \\ & & & 1 & 3 \end{bmatrix}.$$

2. Prescribed convergence for Arnoldi's method

Thus these Ritz values are generated by the Arnoldi method applied to

$$A = V \,\mathrm{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j) \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 \\ & 1 & 3 & -1 & 0 \\ & & 1 & 3 & 5 \\ & & & 1 & 3 \end{bmatrix} \mathrm{diag}\,(1, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi_{j=1}^{n-1}\sigma_j)^{-1} V^*,$$

with initial vector $b = Ve_1$ and for any unitary V and positive values $\sigma_1, \ldots, \sigma_{n-1}$.

This is not a highly non-normal example, for instance with $\sigma_i \equiv 1$:

$$\|A\|\|A^{-1}\| = 9.7137,$$

and the eigenvector basis W of A has condition number

$$\|W\|\|W^{-1}\| = 4.8003.$$

2. Prescribed convergence for Arnoldi's method

Counterintuitive example 2: We can prescribe the "diverging" Ritz values

$$\begin{aligned}\mathcal{R} = \{ \quad & 1, \\ & (0,2)\,, \\ & (-1,1,3)\,, \\ & (-2,0,2,4)\,, \\ & (1,1,1,1,1)\},\end{aligned}$$

with $$H(\mathcal{R}) = U(\mathcal{S})^{-1} C^{(5)} U(\mathcal{S}) = \begin{bmatrix} 1 & 1 & 0 & -3 & 0 \\ 1 & 1 & 3 & 0 & -31 \\ & 1 & 1 & 6 & 0 \\ & & 1 & 1 & -10 \\ & & & 1 & 1 \end{bmatrix}.$$

They are generated by Arnoldi applied to $A = VH(\mathcal{R})V^*$, $b = Ve_1$ for unitary V.

2. Prescribed convergence for Arnoldi's method

The same "diverging" Ritz values are generated with the exponentially decreasing values 2^{-1}, 2^{-2}, 2^{-3} and 2^{-4} on the subdiagonal of the Hessenberg matrix:

$$A = V \begin{bmatrix} 1 & 2 & 0 & -192 & 0 \\ 0.5 & 1 & 12 & 0 & -15872 \\ & 0.25 & 1 & 48 & 0 \\ & & 0.125 & 1 & -160 \\ & & & 0.0625 & 1 \end{bmatrix} V^*, \qquad b = \|b\| V e_1.$$

Then the rounded residual norms $\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k}|e_k^T y|$ seem to indicate convergence:

$$\{ \quad \tfrac{1}{2}, \quad (0.1118, 0.1118), \quad (0.011, 0.0052, 0.011), \quad (0.0006, 0.0001, 0.0001, 0.0006) \}.$$

3. Prescribed convergence for Arnoldi *and* GMRES

Starting with initial guess $x_0 = 0$, GMRES iterates x_k minimize the residual norm,

$$\|b - Ax_k\| = \min \|b - As\| \quad \text{over all} \quad s \in \mathcal{K}_k(A, b).$$

Writing x_k in the Arnoldi basis,

$$x_k = V_k y_k \in \mathcal{K}_k(A, r_0),$$

and using the Arnoldi decomposition $AV_k = V_{k+1}\tilde{H}_k$, we see that the residual norm is

$$\begin{aligned} \|b - Ax_k\| &= \|b - AV_k y_k\| = \|V_{k+1}\|b\|e_1 - AV_k y_k\| \\ &= \|V_{k+1}(\|b\|e_1 - \tilde{H}_k y_k)\| = \min_{y \in \mathbb{C}^k} \|\|b\|e_1 - \tilde{H}_k y\|. \end{aligned}$$

Thus the residual norms generated by the GMRES method are fully determined by the Hessenberg matrix $\tilde{H}_k$ and $\|b\|$.

3. Prescribed convergence for Arnoldi *and* GMRES

- We have seen that the subdiagonal entries of $\tilde{H}_k$ can be chosen arbitrarily, for any prescribed Ritz values in the kth iteration.
- Hence there is a chance we can modify the behavior of GMRES while maintaining the prescribed Ritz values.

Example from earlier: Consider the prescribed 'diverging' Ritz values

$$\begin{aligned}\mathcal{R} = \{ \quad & 1, \\ & (0, 2)\,, \\ & (-1, 1, 3)\,, \\ & (-2, 0, 2, 4)\,, \\ & (1, 1, 1, 1, 1)\}\,,\end{aligned}$$

and the prescribed subdiagonal entries of the generated Hessenberg matrix

$$\sigma_1 = 2^{-1}, \quad \sigma_2 = 2^{-2}, \quad \sigma_3 = 2^{-3}, \quad \sigma_4 = 2^{-4}.$$

3. Prescribed convergence for Arnoldi *and* GMRES

The corresponding GMRES convergence curve is

$$\|r^{(0)}\| = 1, \quad \|r^{(1)}\| = \sqrt{\frac{1}{5}}, \quad \|r^{(2)}\| = \sqrt{\frac{1}{5}}, \quad \|r^{(3)}\| = 0.0052, \quad \|r^{(4)}\| = 0.0052.$$

Question: Can we force any GMRES convergence speed with arbitrary Ritz values by modifying the subdiagonal entries?

Not any, because there is a relation between GMRES stagnation and zero Ritz values: A singular Hessenberg matrix corresponds to stagnation in the parallel GMRES process, see [Brown - 1991]. In our example we have

$$\begin{aligned}
\rho_1^{(1)} = 1, \quad \|r^{(1)}\| &= \frac{1}{\sqrt{5}} \\
(\rho_1^{(2)}, \rho_2^{(2)}) = (0, 2), \quad \|r^{(2)}\| &= \frac{1}{\sqrt{5}} \\
(\rho_1^{(3)}, \rho_2^{(3)}, \rho_3^{(3)}) = (-1, 1, 3), \quad \|r^{(3)}\| &= 0.0052 \\
(\rho_1^{(4)}, \rho_2^{(4)}, \rho_3^{(4)}, \rho_4^{(4)}) = (-2, 0, 2, 4), \quad \|r^{(4)}\| &= 0.0052.
\end{aligned}$$

3. Prescribed convergence for Arnoldi *and* GMRES

However, this is the *only* restriction Ritz values put on GMRES residual norms:

Theorem 3 [DT, Meurant - 2012]. Consider a set of tuples of complex numbers

$$\begin{aligned}\mathcal{R} = \{ \quad & \rho_1^{(1)}, \\ & (\rho_1^{(2)}, \rho_2^{(2)}), \\ & \vdots \\ & (\rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}), \\ & (\lambda_1, \ldots\ldots\ldots, \lambda_n)\},\end{aligned}$$

such that $(\lambda_1, \ldots, \lambda_n)$ contains no zero number and n positive numbers

$$f(0) \geq f(1) \geq \cdots \geq f(n-1) > 0,$$

such that the k-tuple $(\rho_1^{(k)}, \ldots, \rho_k^{(k)})$ contains a zero number if and only if

$$f(k-1) = f(k).$$

3. Prescribed convergence for Arnoldi *and* GMRES

Let A be a square matrix of size n and let b be a nonzero n-dimensional vector.
The following assertions are equivalent:

1. The GMRES method applied to A and right-hand side b with zero initial guess yields residuals $r^{(k)},\ k = 0, \ldots, n-1$ such that

$$\|r^{(k)}\| = f(k), \quad k = 0, \ldots, n-1,$$

A has eigenvalues

$$\lambda_1, \ldots, \lambda_n,$$

and

$$\rho_1^{(k)}, \ldots, \rho_k^{(k)}$$

are the Ritz values generated at the kth iteration for $k = 1, \ldots, n-1$.

3. Prescribed convergence for Arnoldi *and* GMRES

2. The matrix A and right hand side b are of the form

$$A = V\,\mathrm{diag}(f(0), D_c^{-*})\,U(\mathcal{S})^{-1}C^{(n)}U(\mathcal{S})\,\mathrm{diag}(f(0)^{-1}, D_c^{*})V^*, \qquad b = \|b\|Ve_1,$$

where V is a unitary matrix, $U(\mathcal{S})$ is the Ritz value companion transform for $\mathcal{R}$ and $C^{(n)}$ is the companion matrix of the polynomial with roots $\lambda_1, \ldots, \lambda_n$. D_c is a nonsingular diagonal matrix such that

$$R_h^{-T}\hat{h} = -f(0)^2 D_c c,$$

$$\hat{h} = [\eta_1, \ldots, \eta_{n-1}]^T, \qquad \eta_k = (f(k-1)^2 - f(k)^2)^{1/2},$$

R_h being the upper triangular factor of the Cholesky decomposition

$$R_h^T R_h = I - \frac{\hat{h}\hat{h}^T}{f(0)^2},$$

and c is the first row of $U(\mathcal{S})$ without its diagonal entry.

Note we exhausted all freedom modulo unitary transformation.

3. Prescribed convergence for Arnoldi *and* GMRES

Example: Standardly converging Ritz values and 'nearly stagnating' GMRES:

$$\begin{aligned}\mathcal{R} = \{ \quad & 5, \\ & (1,5)\,, \\ & (1,4,5)\,, \\ & (1,3,4,5)\,, \\ & (1,2,3,4,5)\}\,,\end{aligned}$$

$\|r^{(0)}\| = 1, \quad \|r^{(1)}\| = 0.9, \quad \|r^{(2)}\| = 0.8, \quad \|r^{(3)}\| = 0.7, \quad \|r^{(4)}\| = 0.6, \quad \|r^{(5)}\| = 0$ gives

$$A = V \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 10.3237 & 1 & 0 & 0 & 0 \\ & 0.8458 & 4 & 0 & 0 \\ & & 3.312 & 3 & 0 \\ & & & 2.4169 & 2 \end{bmatrix} V^*, \qquad b = \|b\| V e_1 .$$

3. Prescribed convergence for Arnoldi *and* GMRES

Again, this is not a highly non-normal example:

$$\|A\|\|A^{-1}\| = 28.9498,$$

and the eigenvector basis W of A has condition number

$$\|W\|\|W^{-1}\| = 57.735.$$

The residual norms $\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k}|e_k^T y|$ for the Ritz pairs are

$$10.3237,$$

$$(0.8458, 0.7886)\,,$$

$$(0.8987, 3.312, 2.0509)\,,$$

$$(0.9906, 2.4169, 2.3137, 1.7303)\ .$$

respectively, i.e. they give misleading information.

Conclusions and future work:

- There is no interlacing property for the Hessenberg matrices in the Arnoldi method.
- The Ritz values generated in the Arnoldi method can behave arbitrarily badly.
- Convergence of Ritz values need not say anything about the behavior of GMRES residual norms (zero Ritz values excepted). For close to normal matrices, the opposite has been suggested [van der Vorst, Vuik - 1993].
- Extension to harmonic Ritz values which determine the GMRES polynomials?
- It is desirable to have similar results for popular restarted versions of GMRES and the Arnoldi method (see e.g. the failure of restarted Arnoldi with exact shifts explained in [Embree - 2009]).

For more details see: DUINTJER TEBBENS J, MEURANT G: *Any Ritz value behavior is possible for Arnoldi and for GMRES*, SIMAX vol. 33 no. 3, pp 958–978, 2012, available at www.cs.cas.cz/duintjertebbens/duintjertebbens_pub.html

Thank you for your attention!

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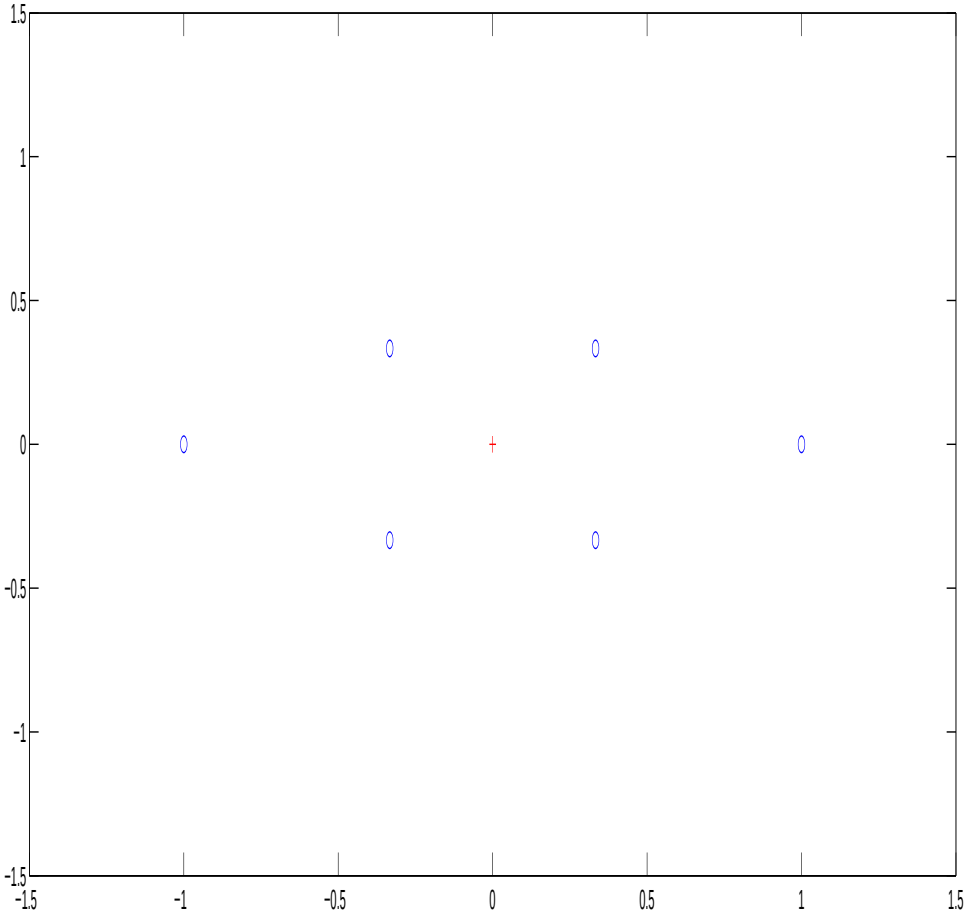
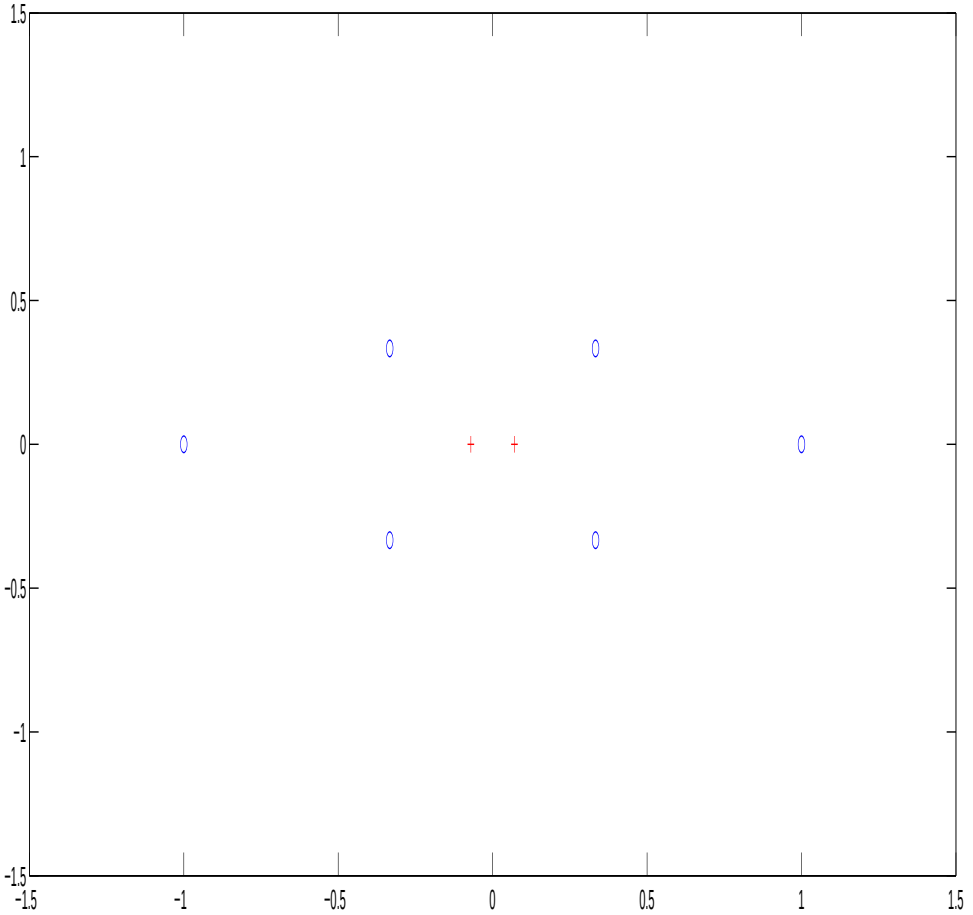
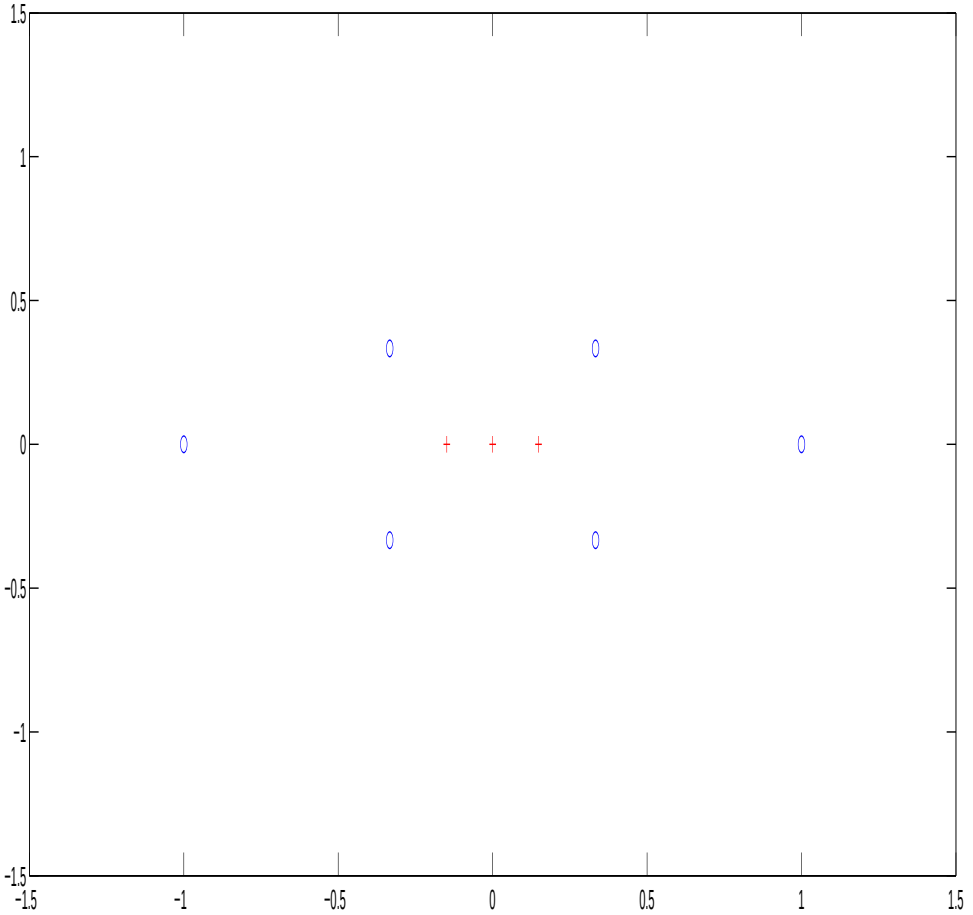
Counterintuitive normal example



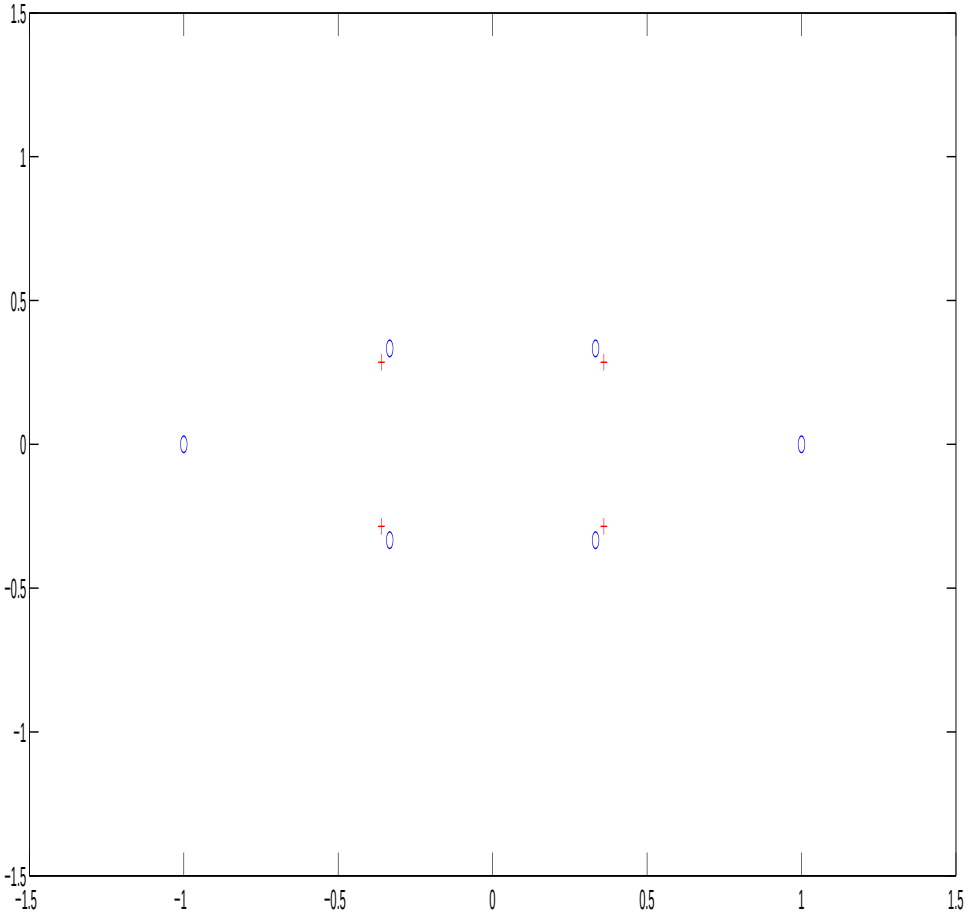
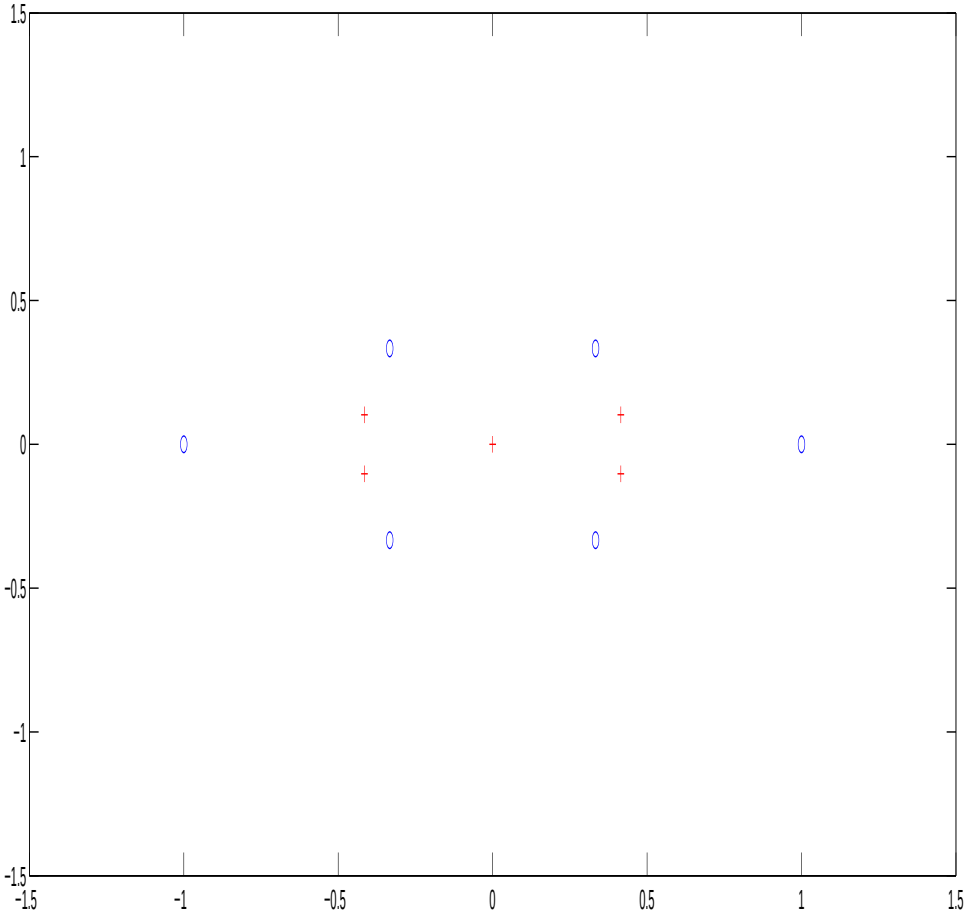
Counterintuitive normal example



Counterintuitive normal example



Counterintuitive normal example



Counterintuitive normal example