Abstract Algebraic Logic – 4th lesson

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Abstract Algebraic Logic

AAL is the evolution of Algebraic Logic that wants to:

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

Abstract Algebraic Logic

What have we done so far?

- understand the several ways by which a logic can be given an algebraic semantics
- build a general and abstract theory of non-classical logics based on their relation to algebras
- understand the rôle of connectives in (non-)classical logics: implication, equivalence, disjunction,...
- classify non-classical logics
- find general results connecting logical and algebraic properties (bridge theorems)
- generalize properties from syntax to semantics (transfer theorems)
- advance the study of particular (families of) non-classical logics by using the abstract notions and results

Bridge theorems vs. transfer theorems

Theorem 4.1 (Bloom)

Let L *be a logic. Then:* $\mathbf{P}_{\mathrm{U}}(\mathbf{MOD}(\mathrm{L})) = \mathbf{MOD}(\mathrm{L})$ *iff* L *is finitary.*

It is a brigde theorem, relating a logical property with an algebraic (or matricial) one.

Theorem 4.2

Given a logic L *in a language* $\mathcal{L}$*, the following conditions are equivalent:*

1. L *is finitary, i.e.* Th_{L} *is a finitary closure operator.*
2. $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}$ *is a finitary closure operator for any* $\mathcal{L}$*-algebra* $\boldsymbol{A}$*.*

It is a transfer theorem, transfering a property of $\boldsymbol{Fm}_{\mathcal{L}}$ to a formally equal property of all $\mathcal{L}$-algebras.

Deduction theorems – 1

A logic L has the parameterized local deduction-detachment theorem if there is a family of sets of formulae $\Sigma \subseteq \mathcal{P}(Fm_{\mathcal{L}})$ in two variables (and possible parameters) such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$,

$\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ iff $\exists \Delta(x, y, \overrightarrow{z}) \in \Sigma$ such that $\Gamma \vdash_{\mathrm{L}} \bigcup_{\overrightarrow{\gamma} \in Fm_{\mathcal{L}}} \Delta(\varphi, \psi, \overrightarrow{\gamma})$.

Theorem 4.3

A logic L is protoalgebraic iff it has the parameterized local deduction-detachment theorem.

Deduction theorems – 2

A logic L has the local deduction-detachment theorem (LDDT) if it has the parameterized local deduction-detachment theorem with an empty set of parameters, i.e. there is a family of sets of formulae $\Sigma \subseteq \mathcal{P}(Fm_{\mathcal{L}})$ in two variables such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \text{ iff } \exists \Delta(x, y) \in \Sigma \text{ such that } \Gamma \vdash_{\mathrm{L}} \Delta(\varphi, \psi).$$

Logic	Σ
Ł (infinitely-valued Łukasiewicz logic)	$\{p \to^n q \mid n \geq 0\}$
global modal logic T	$\{\Box^n p \to q \mid n \geq 0\}$

Deduction theorems – 3

A class of models of a logic $\mathbb{K} \subseteq \mathbf{MOD}(\mathrm{L})$ has the L-filter-extension-property iff for all $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{B}, G\rangle \in \mathbb{K}$ such that $\langle \boldsymbol{A}, F\rangle \subseteq \langle \boldsymbol{B}, G\rangle$ and every $F' \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ such $F \subseteq F'$ and $\langle \boldsymbol{A}, F'\rangle \in \mathbb{K}$, there exists a $G' \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$ such that $G \subseteq G'$, $\langle \boldsymbol{B}, G'\rangle \in \mathbb{K}$, and $G' \cap A = F'$.

Theorem 4.4 (Czelakowski, Blok-Pigozzi)

Let L *be a finitary protoalgebraic logic. TFAE:*

1. L *has the LDDT.*
2. $\mathbf{MOD}(\mathrm{L})$ *has the* L*-filter-extension-property.*
3. $\mathbf{MOD}^*(\mathrm{L})$ *has the* L*-filter-extension-property.*

Deduction theorems – 4

A logic L has the global deduction-detachment theorem (GDDT) if it has the local deduction-detachment theorem with a set Σ consisting of just one finite set of formulae i.e. there is a finite $\Delta(x, y) \subseteq Fm_{\mathcal{L}}$ in two variables such that for all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \text{ iff } \Gamma \vdash_{\mathrm{L}} \Delta(\varphi, \psi).$$

Logic	Δ
CL, IL, local modal logics	$\{p \to q\}$
$Ł_n$ (n-valued Łukasiewicz logic)	$\{p \to^n q\}$
global $S4$ and $S5$	$\{\Box p \to q\}$

Deduction theorems – 5

A class of models of a logic $\mathbb{K} \subseteq \mathbf{MOD}(\mathrm{L})$ has formula-definable principal L-filters if there is a finite set of formulae $\Delta(x, y) = \{\delta_i(x, y) \mid i < n\}$ of formulae in two variables such that, for every $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and every $a \in A$,

$$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup \{a\}) = \{b \in A \mid \forall \delta \in \Delta, \delta^{\boldsymbol{A}}(a, b) \in F\}.$$

Theorem 4.5 (Blok-Pigozzi)

Let L be a finitary protoalgebraic logic. TFAE:

1. L *has the GDDT.*
2. $\mathbf{MOD}(\mathrm{L})$ *has formula-definable principal* L*-filters.*
3. $\mathbf{MOD}^*(\mathrm{L})$ *has formula-definable principal* L*-filters.*

Deduction theorems – 6

A dual Brouwerian semilattice is an algebra $\boldsymbol{A} = \langle A, *^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ such that $\langle A, \vee^{\boldsymbol{A}}, \top^{\boldsymbol{A}} \rangle$ is a bounded join-semilattice and, for $a, b \in A$, there exists $a *^{\boldsymbol{A}} b$, the smallest element c such that $a \leq b \vee^{\boldsymbol{A}} c$. Hence for every $a, b, c \in A$:

$$a *^{\boldsymbol{A}} b \leq c \text{ iff } a \leq b \vee^{\boldsymbol{A}} c.$$

Theorem 4.6 (Czelakowski)

Let L be a finitary protoalgebraic logic. TFAE:

1. L *has the GDDT.*
2. *The join-semilattice of finitely axiomatizable theories of* L *is dually Brouwerian.*
3. *For every* $\boldsymbol{A}$*, the join-semilattice of finitely generated* L*-filters of* $\boldsymbol{A}$ *is dually Brouwerian.*

Deduction theorems – 7

A quasivariety $\mathbb{K}$ has equationally definable principal relative congruences (EDPRC) if there is a finite set of equations in at most four variables $\{\varepsilon_i(x_0, x_1, y_0, y_1) \approx \delta_i(x_0, x_1, y_0, y_1) \mid i < n\}$ such that for every algebra $\boldsymbol{A} \in \mathbb{K}$ and all $a, b, c, d \in A$,

$$\langle c, d\rangle \in \Theta^{\boldsymbol{A}}_{\mathbb{K}}(a, b) \text{ iff } \forall i < n\ \varepsilon^{\boldsymbol{A}}_i(a, b, c, d) = \delta^{\boldsymbol{A}}_i(a, b, c, d),$$

where $\Theta^{\boldsymbol{A}}_{\mathbb{K}}(a, b)$ denotes the relative congruence generated by $\langle a, b\rangle$.

Theorem 4.7 (Blok-Pigozzi)

Let L *be a finitary and finitely algebraizable logic. TFAE:*

1. L *has the GDDT.*
2. $\mathbf{ALG}^*(\mathrm{L})$ *has EDPRC.*

Deduction theorems – 8

A quasivariety $\mathbb{K}$ has the relative congruence extension property (RCEP) if, only if, for every $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{K}$ such that $\boldsymbol{B} \subseteq \boldsymbol{A}$ and every $\theta \in \boldsymbol{Con}_{\mathbb{K}}(\boldsymbol{B})$, there exists $\theta' \in \boldsymbol{Con}_{\mathbb{K}}(\boldsymbol{A})$ such that $\theta' \cap B^2 = \theta$.

Theorem 4.8 (Blok-Pigozzi, Czelakowski-Dziobiak)

Let L *be a finitary and finitely algebraizable logic. TFAE:*

1. L *has the LDDT.*
2. $\mathbf{ALG}^*(\mathrm{L})$ *has the RCEP.*

Beth property – 1

Let L be a logic and $P, R \subseteq Var$, $P \cap R = \emptyset$, $\Gamma(\overrightarrow{p}, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$, $\overrightarrow{p} \in P$, $\overrightarrow{r} \in R$. We say that $\Gamma(\overrightarrow{p}, \overrightarrow{r})$ defines R explicitly in terms of P if for every $r \in R$ there is $\varphi_r \in Fm_{\mathcal{L}}$ with variables in P such that $\langle r, \varphi_r \rangle \in \Omega(\mathrm{Fi}_{\mathrm{L}}^{P \cup R}(\Gamma))$ (filter generated in the subalgebra of formulae in variables $P \cup R$).

We say that $\Gamma(\overrightarrow{p}, \overrightarrow{r})$ defines R implicitly in terms of P if for every $R' \subseteq Var$, $R' \cap (P \cup R) = \emptyset$, $|R'| = |R|$, and every bijection f between R and R', we have that for every $r \in R$, $\langle r, f(r) \rangle \in \Omega(\mathrm{Fi}_{\mathrm{L}}^{P \cup R \cup R'}(\Gamma))$.

L has the Beth property if for all disjoint sets of variables P and R, each set $\Gamma(\overrightarrow{p}, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$ that defines R implicitly in terms of P, defines also R explicitly in terms of P.

Beth property – 2

Let $\mathbb{K}$ be a class of algebras of the same type, $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{K}$, and $h \colon \boldsymbol{A} \to \boldsymbol{B}$ a homomorphism. h is an epimorphism in $\mathbb{K}$ if for every $\boldsymbol{C} \in \mathbb{K}$ and each $g, g' \colon \boldsymbol{B} \to \boldsymbol{C}$, if $g \circ h = g' \circ h$, then $g = g'$.

A class $\mathbb{K}$ of algebras has the property that epimorphisms are surjective (ES) if every epimorphism between algebras of $\mathbb{K}$ is a surjective mapping.

Theorem 4.9 (Hoogland)

Let L *be an algebraizable logic. TFAE:*

1. L *has the Beth property.*
2. $\mathbf{ALG}^*(\mathrm{L})$ *has the ES.*

Craig interpolation

A logic L has the Craig interpolation property for consequence if for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$, there is $\Gamma' \subseteq Fm_{\mathcal{L}}$ with variables in $Var(\Gamma) \cap Var(\varphi)$ such that $\Gamma \vdash_{\mathrm{L}} \Gamma'$ and $\Gamma' \vdash_{\mathrm{L}} \varphi$.

A class of algebras $\mathbb{K}$ has the amalgamation property if for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{K}$ and any embeddings $f \colon \boldsymbol{C} \to \boldsymbol{A}$ and $g \colon \boldsymbol{C} \to \boldsymbol{B}$, there is $\boldsymbol{D} \in \mathbb{K}$ and embeddings $h \colon \boldsymbol{A} \to \boldsymbol{D}$ and $t \colon \boldsymbol{B} \to \boldsymbol{D}$ such that $h \circ f = t \circ g$.

Theorem 4.10 (Czelakowski)

Let L *be an algebraizable logic with GDDT. TFAE:*

1. L *has the Craig interpolation property for consequence.*
2. $\mathbf{ALG}^*(\mathrm{L})$ *has the amalgamation property.*

A non-protoalgebraic logic – 1

$\mathrm{CPC}_{\wedge\vee}$ is defined as the $\{\wedge, \vee\}$-fragment of classical logic.

Gentzen presentation [Font and Verdú, 1991]

Hilbert presentation [Dyrda and Prucnal, 1980]:

$\varphi \wedge \psi \rhd \varphi$	$\varphi \vee (\psi \vee \xi) \rhd (\varphi \vee \psi) \vee \xi$
$\varphi \wedge \psi \rhd \psi \wedge \varphi$	$(\varphi \vee \psi) \vee \xi \rhd \varphi \vee (\psi \vee \xi)$
$\varphi, \psi \rhd \varphi \wedge \psi$	$\varphi \vee (\psi \wedge \xi) \rhd (\varphi \vee \psi) \wedge (\varphi \vee \xi)$
$\varphi \rhd \varphi \vee \psi$	$(\varphi \vee \psi) \wedge (\varphi \vee \xi) \rhd \varphi \vee (\psi \wedge \xi)$
$\varphi \vee \psi \rhd \psi \vee \varphi$	$\varphi \wedge (\psi \vee \xi) \rhd (\varphi \wedge \psi) \vee (\varphi \wedge \xi)$
$\varphi \vee (\varphi \vee \psi) \rhd \varphi \vee \psi$	$\varphi \vee \varphi \rhd \varphi$

It is a logic without theorems, not almost inconsistent, and hence not protoalgebraic.

A non-protoalgebraic logic – 2

$\mathbf{2}_{\wedge,\vee}$: $\{\wedge, \vee\}$-reduct of the two-element Boolean algebra $\mathbf{2}$

$\mathrm{CPC}_{\wedge\vee} = \models_{\mathbf{2}_{\wedge,\vee}}$

$\mathbf{V}(\mathbf{2}_{\wedge,\vee}) = \mathbb{D}$ (variety of distributive lattices)

Is $\mathbb{D}$ the algebraic semantics of $\mathrm{CPC}_{\wedge\vee}$?

A non-protoalgebraic logic – 3

Theorem 4.11

$\mathbf{ALG}^*(\mathrm{CPC}_{\wedge\vee}) = \{\boldsymbol{A} \in \mathbb{D} \mid \boldsymbol{A}$ *has a maximum element* 1 *and for every* $a, b \in A$ *if* $a < b$ *then there is* $c \in A$ *such that* $a \vee c \neq 1$ *and* $b \vee c = 1\}$ *(a proper subclass of* $\mathbb{D}$*, not even quasivariety).*

Theorem 4.12

$\mathbb{D}$ is not the equivalent algebraic semantics of any algebraizable logic.

$\mathbf{ALG}(\mathrm{CPC}_{\wedge\vee}) = \mathbb{D}$ [Font-Jansana] (alternative AAL theory based on generalized models)

- $\mathbf{ALG}(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{ALG}^*(\mathrm{L}))$.
- If L is protoalgebraic, then $\mathbf{ALG}(\mathrm{L}) = \mathbf{ALG}^*(\mathrm{L})$.

Frege hierarchy – 1

Proposition 4.13

A logic L *in a language* $\mathcal{L}$ *is protoalgebraic iff for every* $T \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$

$$\langle \alpha, \beta \rangle \in \Omega_{\boldsymbol{Fm}_{\mathcal{L}}}(T) \text{ implies } \mathrm{Th_L}(T, \alpha) = \mathrm{Th_L}(T, \beta).$$

Frege relation: $\langle \varphi, \psi \rangle \in \Lambda_{\mathrm{L}}$ iff $\varphi \vdash_{\mathrm{L}} \psi$ and $\psi \vdash_{\mathrm{L}} \varphi$.

Selfextensional logic: L is selfextensional iff $\Lambda_{\mathrm{L}} \in \boldsymbol{Con}(\boldsymbol{Fm}_{\mathcal{L}})$.

Frege relation w.r.t. a theory: $\langle \varphi, \psi \rangle \in \Lambda_{\mathrm{L}}(T)$ iff $T, \varphi \vdash_{\mathrm{L}} \psi$ and $T, \psi \vdash_{\mathrm{L}} \varphi$.

Fregean logic: L is Fregean iff $\Lambda_{\mathrm{L}}(T) \in \boldsymbol{Con}(\boldsymbol{Fm}_{\mathcal{L}})$ for every $T \in \mathrm{Th}(\mathrm{L})$.

Frege hierarchy – 2

Inc, AInc, CL, IL, $\mathrm{CPC}_{\wedge\vee}$ are Fregean.

Dumb is selfextensional but not Fregean.

Ł_3 is not selfextensional ($\varphi \dashv\vdash \psi$ does not imply $\neg\varphi \dashv\vdash \neg\psi$; take $\varphi = p$ and $\psi = \neg(p \to \neg p)$, $e(p) = \frac{1}{2}$).

Theorem 4.14

- *Every protoalgebraic Fregean logic with theorems is regularly algebraizable.*
- *Every finitary and protoalgebraic Fregean logic with theorems is regularly, finitely algebraizable.*

Linear logic is not Fregean.

Infinitely-valued Łukasiewicz logics

$\boldsymbol{A} = \langle [0,1], \to, \neg \rangle$, $a \to b = \min\{1, 1-a+b\}$ and $\neg a = 1-a$.

- Infinitary version $Ł_\infty$: $\models_{\langle \boldsymbol{A}, \{1\} \rangle}$
- Finitary version Ł: finitary companion of $Ł_\infty$
  $\Gamma \vdash_{Ł} \varphi$ iff there is a finite $\Gamma_0 \subseteq \Gamma$ s.t. $\Gamma_0 \models_{\langle \boldsymbol{A}, \{1\} \rangle} \varphi$.
- Degree-preserving version $Ł^{\leq}$: $\varphi_1, \ldots, \varphi_n \vdash_{Ł^{\leq}} \varphi$ iff for each $\boldsymbol{A}$-evaluation e, $\min\{e(\varphi_1), \ldots, e(\varphi_n)\} \leq e(\varphi)$.

They all have the same theorems.

$Ł_\infty$ is Rasiowa-implicative (but $\mathbf{ALG}^*(Ł_\infty)$ is not quasivariety) and not selfextensional (counterexample as in $Ł_3$).

Ł is Rasiowa-implicative (and strongly BP-algebraizable) and not selfextensional (counterexample as in $Ł_3$).

$Ł^{\leq}$ is selfextensional (not Fregean) and not protoalgebraic.

Disjunction in Classical Logic

(PD) $\varphi \vdash_{\mathrm{CL}} \varphi \vee \psi$ and $\psi \vdash_{\mathrm{CL}} \varphi \vee \psi$

PCP If $\Gamma, \varphi \vdash_{\mathrm{CL}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CL}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CL}} \chi$.

The same holds for many other logics: IL, Ł, FL_{ew}, HL, ...

(PD) and PCP could be equivalently formulated as:

$\Gamma, \varphi \vdash_{\mathrm{CL}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CL}} \chi$, if and only if, $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CL}} \chi$.

Dummett in '*The Logical Basis of Metaphysics*, HUP, 1991' says about (a weaker variant of) PCP:

> *If this law does not hold, the operator $\vee$ could not legitimately be called disjunction operator.*

A problem

Theorem 4.15

In FL_e, the lattice connective $\vee$ does not satisfy the PCP (it would entail $\varphi \vee \psi \vdash (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$).

A solution of this problem:

Theorem 4.16

The connective $\vee'$ defined as $\varphi \vee' \psi = (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$ satisfies

(PD) $\varphi \vdash (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$ *and* $\psi \vdash (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$

PCP *If $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$, then $\Gamma, (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1}) \vdash \chi$.*

A bigger problem

Theorem 4.17

In the implication fragment of Gödel-Dummett logic we cannot define any connective $\vee$ *satisfying* (PD) *and* PCP*.*

A solution of this problem:

Theorem 4.18

The 'connective' $\{(\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi\}$ *satisfies*

$(\mathrm{PD})_\varphi$ $\quad \varphi \vdash (\varphi \to \psi) \to \psi$ *and* $\varphi \vdash (\psi \to \varphi) \to \varphi$

$(\mathrm{PD})_\psi$ $\quad \psi \vdash (\varphi \to \psi) \to \psi$ *and* $\psi \vdash (\psi \to \varphi) \to \varphi$

PCP $\quad$ *If* $\Gamma, \varphi \vdash \chi$ *and* $\Gamma, \psi \vdash \chi$*, then*

$$\Gamma, (\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi \vdash \chi.$$

An even bigger problem

Theorem 4.19

In FL *no finite set of formulae of two variables defines any 'connective' satisfying* (PD) *and* PCP.

BUT there is still a solution of this problem:

Theorem 4.20

The following 'connective' satisfies both (PD) *and* PCP
$\{\gamma_1(\varphi) \vee \gamma_2(\psi) \mid$ *where* γ_1, γ_2 *are iterated conjugates*$\}$.

An iterated conjugate of φ is a formula $\gamma_{\alpha_1}(\gamma_{\alpha_2}(\ldots\gamma_{\alpha_n}(\varphi)\ldots))$ where $\gamma_{\alpha_i} = \lambda_{\alpha_i}(\varphi) = (\alpha_i \backslash \varphi \& \alpha_i) \wedge \overline{1}$ or $\gamma_{\alpha_i} = \rho_{\alpha_i}(\varphi) = (\alpha_i \& \varphi / \alpha_i) \wedge \overline{1}$ for some formulae α_i.

Definition and useful conventions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulae. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{F}m^{\leq\omega}\}.$$

$$\Sigma_1 \nabla \Sigma_2 = \bigcup \{\varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2\}$$

Generalized disjunctions

A (parameterized) set of formulae ∇ is a (p-)protodisjunction if:

(PD) $\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$ and $\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$

We will consider the following three properties:

wPCP	$\varphi \vdash_{\mathrm{L}} \chi$	and	$\psi \vdash_{\mathrm{L}} \chi$	implies	$\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
PCP	$\Gamma, \varphi \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \psi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$
sPCP	$\Gamma, \Sigma \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \Pi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \Sigma \nabla \Pi \vdash_{\mathrm{L}} \chi$

Clearly: $\mathrm{sPCP} \Rightarrow \mathrm{PCP} \Rightarrow \mathrm{wPCP}$

Theorem 4.21

For finitary logics: $\mathrm{sPCP} \Leftrightarrow \mathrm{PCP} \not\Leftarrow \mathrm{wPCP}$
But in general: $\mathrm{sPCP} \not\Leftarrow \mathrm{PCP}$

We define also transferred variants of these notions.

Example 4.22

Consider the non-distributive lattice *diamond*, with the domain $\{\bot, a, b, t, \top\}$, with t as central element, and the finitary logic given by all matrices over this algebra with a lattice filter.

Observe: $\Gamma \vdash \varphi$ iff $\bigwedge e[\Gamma] \leq e(\varphi)$ for every evaluation e.

$\vee$ is a protodisjunction with wPCP.

Assume now, for a contradiction, that it satisfies the PCP too. Then from $\varphi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and $\chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ we obtain $\varphi \vee \chi, \psi \vdash (\varphi \wedge \psi) \vee \chi$ and thus also (applying the PCP again) $\varphi \vee \chi, \psi \vee \chi \vdash (\varphi \wedge \psi) \vee \chi$ (a form of distributivity). Then, we reach a contradiction by observing that $a \vee b = t \vee b = \top$ while $(a \wedge t) \vee b = \bot \vee b = b$.

Example 4.23

Let $\boldsymbol{A}$ be a complete distributive lattice such that it is not a dual frame, i.e. there are elements $x_i \in A$ for $i \geq 0$ such that

$$\bigwedge_{i \geq 1}(x_0 \vee x_i) \not\leq x_0 \vee \bigwedge_{i \geq 1} x_i$$

expand the lattice language by constants $\{c_i \mid i \geq 0\} \cup \{c\}$ and define algebra $\boldsymbol{A}'$ in this language by setting $c_i^{\boldsymbol{A}'} = x_i$ and $c = \bigwedge_{i \geq 1} x_i$. Then we define the logic L in this language semantically given by the class of matrices $\{\langle \boldsymbol{A}', F\rangle \mid F$ is a principal lattice filter in $\boldsymbol{A}\}$.

Observe: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\bigwedge_{\psi \in \Gamma} e(\psi) \leq e(\varphi)$ for each $\boldsymbol{A}$-evaluation e.

Example 4.24 (continuation)

First we show that $\vee$ enjoys the PCP: assume that for each e evaluation holds $(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\varphi) \leq e(\chi)$ and $(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\psi) \leq e(\chi)$, thus $[(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\varphi)] \vee [(\bigwedge_{\delta\in\Gamma} e(\delta)) \wedge e(\psi)] \leq e(\chi)$, the distributivity of $\boldsymbol{A}$ completes the proof. Finally, by the way of contradiction, assume that $\vee$ enjoys the sPCP. Observe that: $c_0 \vdash_{\mathrm{L}} c_0 \vee c$ and $\{c_i \mid i \geq 1\} \vdash_{\mathrm{L}} c_0 \vee c$. Using the sPCP we obtain $\{c_0 \vee c_i \mid i \geq 1\} \vdash_{\mathrm{L}} c_0 \vee c$—a contradiction.

Syntactical characterization

Theorem 4.25

Let ∇ a *commutative and idempotent* *p-protodisjunction. TFAE:*

1. ∇ *satisfies* sPCP,
2. *whenever* $\Gamma \vdash_{\mathrm{L}} \varphi$ *we have also:* $\Gamma \nabla \chi \vdash_{\mathrm{L}} \varphi \nabla \chi$ *for each* χ.

This theorem was previously known for finitary logics and PCP.

Theorem 4.26

TFAE:

1. *There is a (p-)protodisjunction satisfying* wPCP.
2. *For each (surjective) substitution* σ *and formulae* φ, ψ*:*

$$\mathrm{Th}_{\mathrm{L}}(\sigma\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\sigma\psi) = \mathrm{Th}_{\mathrm{L}}(\sigma[\mathrm{Th}_{\mathrm{L}}(\varphi) \cap \mathrm{Th}_{\mathrm{L}}(\psi)]).$$

If there is a (p-)protodisjunction satisfying wPCP, then $\mathrm{Th}_{\mathrm{L}}(p) \cap \mathrm{Th}_{\mathrm{L}}(q)$ is the largest.

More definitions

$\mathrm{Th}(\mathrm{L})$ is both a closure system and a complete lattice. A theory is intersection-prime if it is finitely $\cap$-irreducible in $\mathrm{Th}(\mathrm{L})$.

Definition 4.27

We say that L:

- is distributive if $\mathrm{Th}(\mathrm{L})$ is a distributive lattice
- is framal if $\mathrm{Th}(\mathrm{L})$ is a frame (meets distribute over arbitrary joins)
- has the IPEP (intersection-prime extension property) if intersection-prime theories form a base of $\mathrm{Th}(\mathrm{L})$, i.e. if $T \in \mathrm{Th}(\mathrm{L})$ and $\varphi \notin T$, there is an intersection-prime theory $T' \supseteq T$ such that $\varphi \notin T'$.

We define filter-distributivity/framality by demanding the defining conditions for $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for each $\mathcal{L}$-algebra $\boldsymbol{A}$.

Finitary vs. IPEP logics

Theorem 4.28

Every finitary logic has IPEP *and NOT vice versa.*

Example 4.29

Recall that $Ł_\infty$. If $T \nvdash_{Ł_\infty} \chi$, then there is an evaluation e such that $e[T] = \{1\}$ and $e(\chi) \neq 1$. We define $T' = e^{-1}[\{1\}]$. Obviously T' is a theory, $T \subseteq T'$ and $T' \nvdash_{Ł_\infty} \chi$. Assume that T' is not intersection-prime; thus there are formulae $\varphi, \psi \notin T'$ such that $T' = \mathrm{Th}_{Ł_\infty}(T, \varphi) \cap \mathrm{Th}_{Ł_\infty}(T, \psi)$. Assume without loss of generality that $e(\varphi) \leq e(\psi)$, so $e(\varphi \to \psi) = 1$ and so $\varphi \to \psi \in T'$. Thus $\psi \in \mathrm{Th}_{Ł_\infty}(T, \varphi)$ (because $\varphi, \varphi \to \psi \vdash_{Ł_\infty} \psi$) and thus $\psi \in T'$—a contradiction. Therefore, it has the IPEP.

Prime theories

Definition 4.30

A theory T is ∇-prime if it is consistent and $T \vdash \varphi \nabla \psi$ implies $T \vdash \varphi$ or $T \vdash \psi$.

∇ has the PEP if ∇-prime theories form a base of $\mathrm{Th}(\mathrm{L})$.

Theorem 4.31

If ∇ has PCP, then ∇-prime and intersection-prime theories coincide.

Theorem 4.32

Let L be a logic satisfying the IPEP. TFAE:

1. *∇ has the sPCP.*
2. *∇ has the PCP.*
3. *∇ has the PEP.*

Theorem 4.33 (Characterizations of sPCP)

The following are equivalent:

1. ∇ *enjoys the* sPCP,
2. ∇ *enjoys the* wPCP *and the logic* L *is framal,*
3. ∇ *enjoys the* wPCP *and the logic* L *is filter-framal,*
4. ∇ *enjoys the transferred* sPCP.

Theorem 4.34 (Characterizations of PCP)

Let L *have* IPEP*. The following are equivalent:*

1. ∇ *enjoys the* PCP,
2. ∇ *enjoys the* wPCP *and the logic* L *is distributive,*
3. ∇ *enjoys the* wPCP *and the logic* L *is filter-distributive,*
4. ∇ *enjoys the transferred* PCP.

Theorem 4.35

Let L *be a protoalgebraic logic.*

- L *is distributive/framal IFF there is a p-protodisjunction* ∇ *which has* $\mathrm{PCP}/\mathrm{sPCP}$.
- *If* L *has* IPEP *and is distributive, then it is filter-framal.*
- *If* ∇ *has* PCP, *then it has transferred* PCP.

Corollary 4.36

Let L *be a logic with the* IPEP, ∇ *a p-protodisjunction with* PCP, *and let* $\mathrm{L}_1, \mathrm{L}_2$ *be axiomatic extensions of* L *by sets of axioms* $\mathcal{A}_1$ *and* $\mathcal{A}_2$, *respectively. Then:*

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \{\varphi \nabla \psi \mid \varphi \in \mathcal{A}_1, \psi \in \mathcal{A}_2\}.$$

Note: we can safely always assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are written in disjoint sets of variables.

Theorem 4.37

Let L *be a logic with the* IPEP, ∇ *a p-protodisjunction with* PCP, *and* $\mathcal{C}$ *a set of positive clauses. Then:*

$$\models_{\{\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L}) \mid \mathbf{A} \models \mathcal{C}\}} = \mathrm{L} + \{\nabla_{\psi \in \Sigma_C} \psi \mid C \in \mathcal{C}\}.$$