# Image Deblurring
# In the Light of DCT

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## The Three Fundamental Challenges of Image Deblurring

 $\longleftrightarrow$ 

1. Incomplete and noisy data (blurred noisy images).
2. Limited accuracy of the solutions, due to ill-posedness.
3. Sheer memory requirements (and associated computing times).

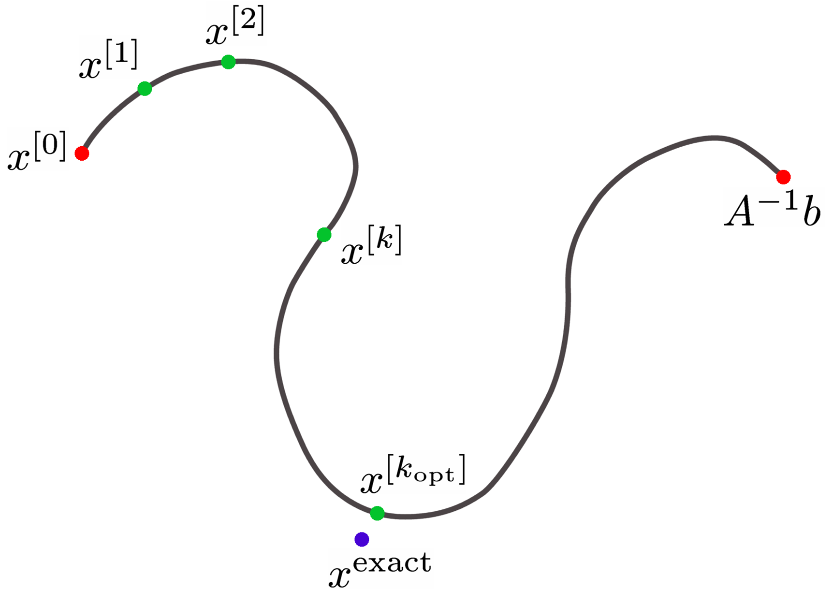
For example, 3-D tomography reconstructions – if solved by naive methods – will require thousands of Gbytes of memory.

Only *new mathematical techniques* can provide more accurate solutions while, at the same time, achieving the necessary substantial reduction in memory demand and computing time.

Several talks at this meeting by Mikula, Renaut, Popa, Rojas, Román, Sima, Voss, . . .

**State-of-the-Art: Iterative Methods + Semi-Convergence**

Large-scale imaging problems can often be solved by iterative methods, such as (preconditioned) regularizing iterations.




During the first iterations, the iterates $x^{[k]}$ tend to be better and better approximations to the exact solution $x^{\rm exact}$.

A some stage they start to diverge and converge towards $A^{-1}b$.

## Goals (and Non-Goals) of This Talk

Our focus is on regularizing iterations associated with the Krylov subspace methods CGLS, MINRES and GMRES.

► The success of these methods is governed by the *initial behavior* of the iterations. We are not concerned with asymptotic analysis.

► We are interested in the *visual quality* of the reconstructions. Error norms and residual norms do not provide enough information.

► We are mainly concerned with *suppression of noise* (from data) in the reconstructions. We are not concerned with noise $\to 0$.

## And Now ... The DCT

We want to do better than just staring at the reconstructions!

Our tool is the **DCT = discrete cosine transformation**, which allows us to study the *spatial frequency contents* in the images.

(DFT could also be used – but we prefer a real transformation.)

Matrix formulation of DCT ($x$ = signal, $\hat{x}$ = spectrum):

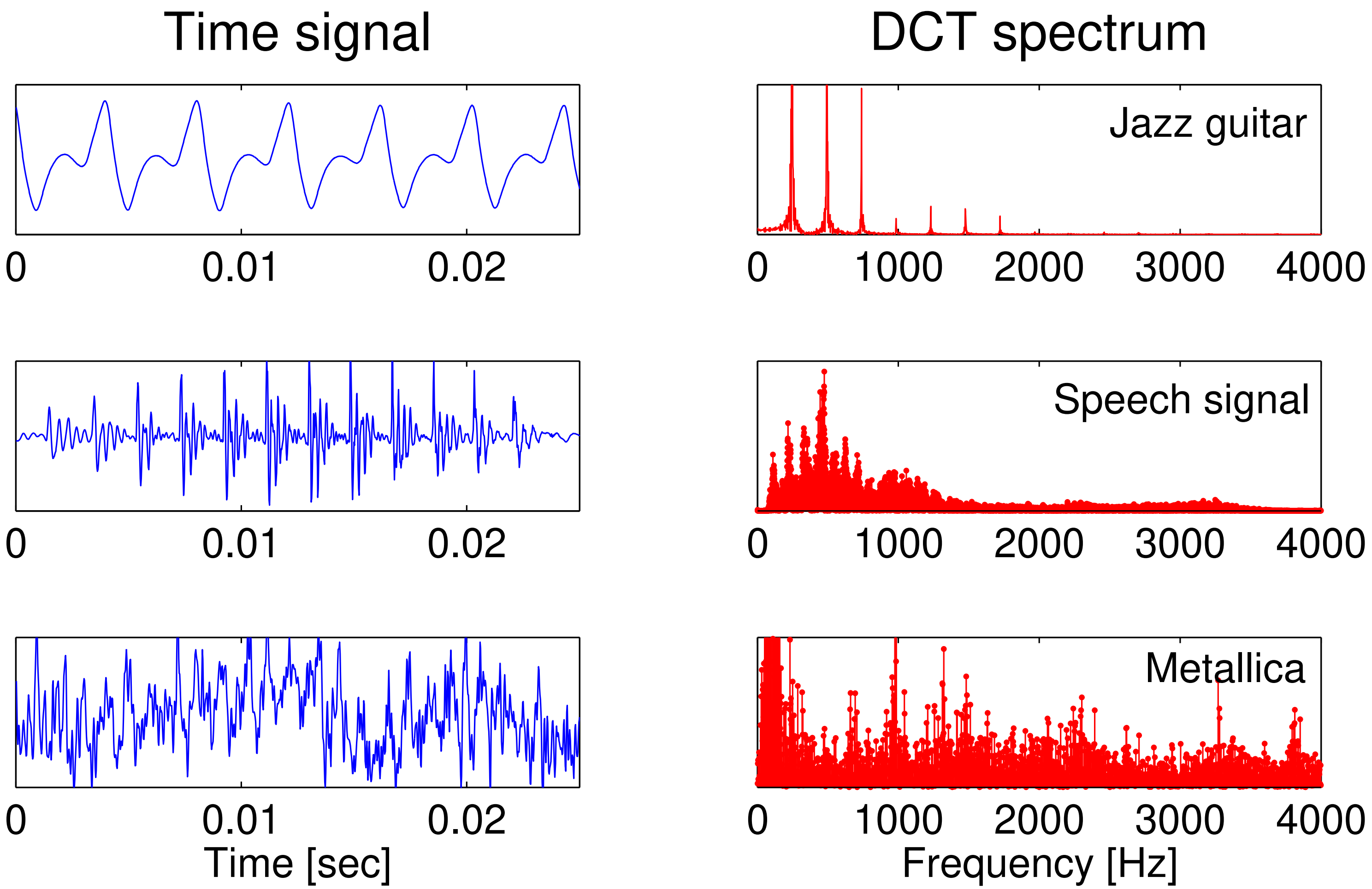
$$\hat{x} = \mathtt{dct}(x) = \mathcal{C}\, x,$$

where $\mathcal{C}$ is an $n \times n$ orthogonal matrix with elements

$$\mathcal{C}_{ij} = \begin{cases} \sqrt{1/n} & i = 1 \\ \sqrt{2/n}\, \cos\bigl(\pi(i-1)(2j-1)/(2n)\bigr), & i > 1. \end{cases}$$

The rows of $\mathcal{C}$ are sampled cosines.

## DCT for Spectral Analysis of Time Series



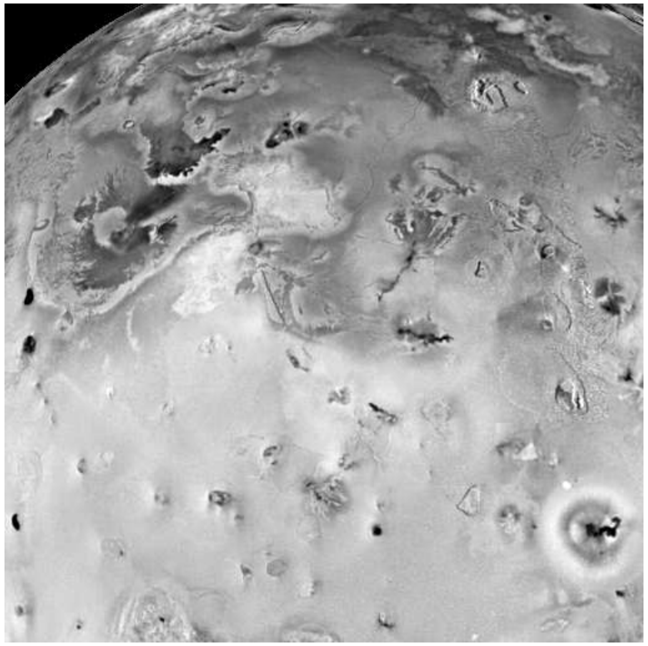

In *white noise* all spectral components have the same expectation.

## Two-Dimensional DCT Analysis

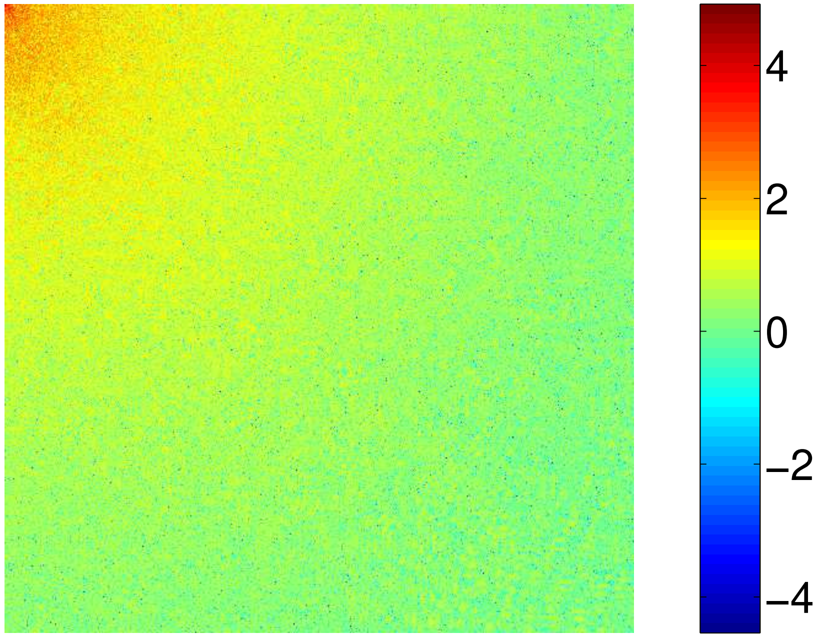
The *two-dimensional DCT* of an image $X$ is:

$$\widehat{X} = \texttt{dct2}(X) = \mathcal{C}_{\mathrm{c}}\, X\, \mathcal{C}_{\mathrm{r}}^T .$$

X = Io (moon of Saturn)



$\log_{10}( \mid C\ X\ C^T \mid )$

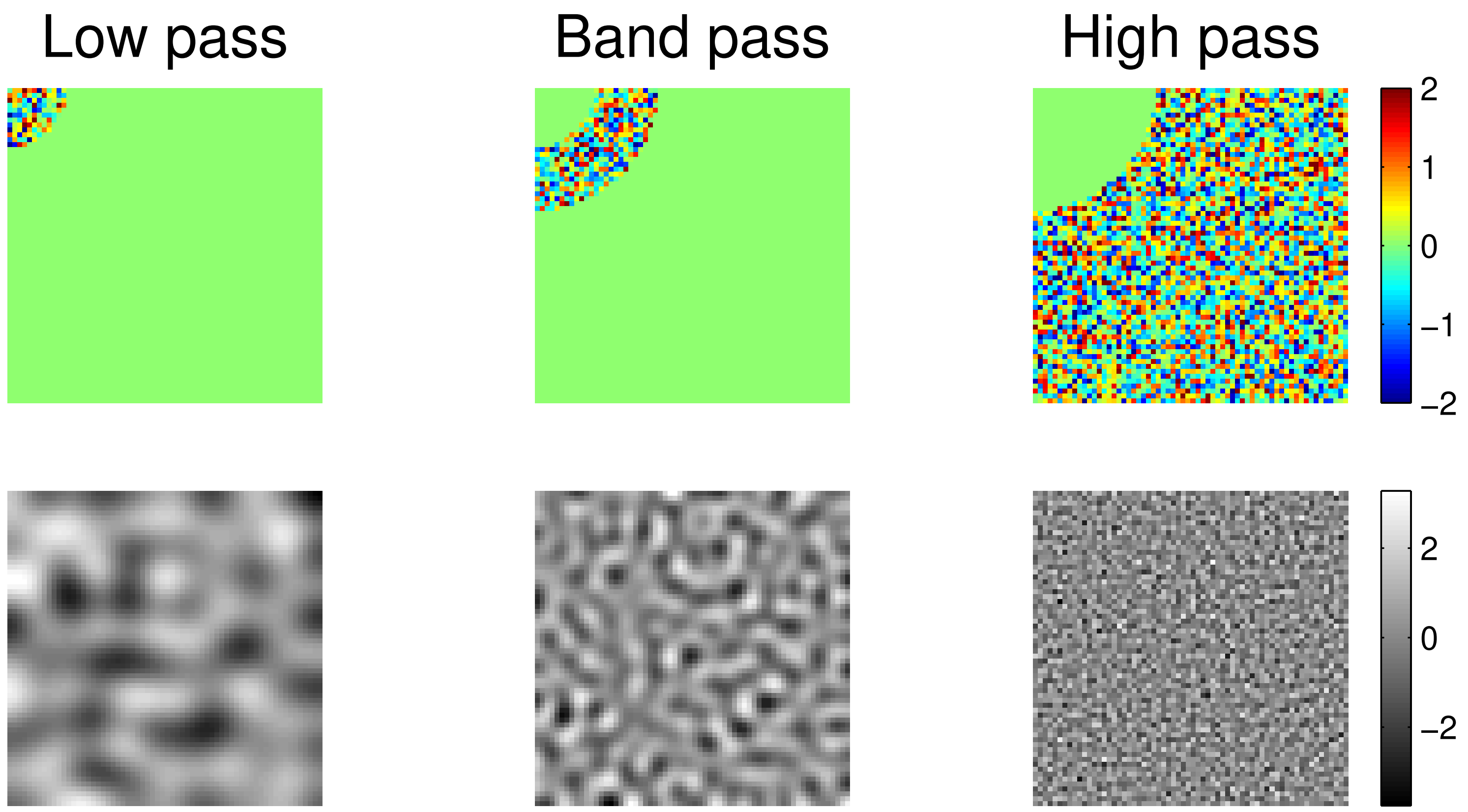



Images are dominated by spatially low-frequent information.

## Filtered White Noise → "Freckles"

$$\Psi = \text{matrix of 0s and 1s}$$

$$X_{\text{filt}} = \texttt{idct2}(\Psi \odot \texttt{randn}(\texttt{n})) \qquad \odot = \text{elementw. mult.}$$



## The Discrete Problem of Image Deblurring

Image deblurring is a discrete ill-posed problem of the form

$$\boxed{A\,x = b}$$

where $x$ and $b$ are "stacked" versions of $X$ and $B$,

$$\boxed{\text{vec notation:} \quad x = \mathrm{vec}\,(X)\,, \quad b = \mathrm{vec}\,(B).}$$

The (huge) PSF matrix $A$ comes from the point spread function and represents the blurring.

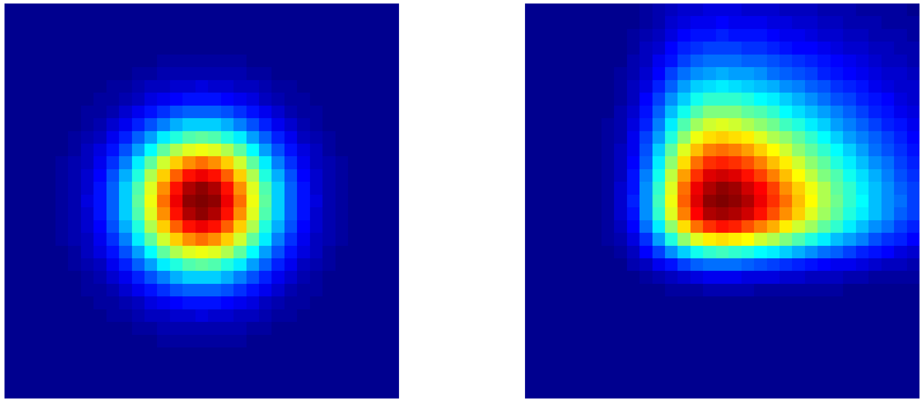
Usually $A$ has structure, e.g., block-Toeplitz with Toeplitz blocks.

The structure depends on the PSF and the boundary conditions.

## Our Test Problems

Sharp and blurred images due to symmetric/nonsymmetric $A$:

Zoom of the corresponding point-spread functions:



## Regularizing Iterations

We consider several Krylov subspace methods of the form

$$\min_x \|A\,x - b\|_2 \quad \text{s.t.} \quad x \in \mathcal{S}_k\,, \qquad\qquad \mathcal{S}_k = \text{Krylov subspace.}$$

**CGLS:**

$$\mathcal{S}_k = \text{span}\left\{A^T b, (A^T A)A^T b, \ldots, (A^T A)^{k-1} A^T b\right\}.$$

**GMRES and MINRES:**

$$\mathcal{S}_k = \text{span}\left\{b, Ab, A^2 b, \ldots, A^{k-1} b\right\}.$$

**RRGMRES and MR-II:**

$$\mathcal{S}_k = \text{span}\left\{Ab, A^2 b, \ldots, A^{k-1} b, A^k b\right\}.$$

Implementations of these methods produce (explicitly or implicitly) an orthonormal basis for $\mathcal{S}_k$.

## Symmetric $A$, Iterations $k = 5$, 10, and 20

## Nonsymmetric $A$, Iterations $k$ = 5, 10, and 20

**A Closer Look at the Krylov Basis – CGLS**

The "power vectors" $(A^T A)^{i-1} A^T b$ of the CGLS Krylov subspace $\mathcal{S}_k$ are just more and more blurred versions of $B$. Not interesting!

$\looparrowright$ We need an orthonormal basis for $\mathcal{S}_k$.

Recall that CGLS is equivalent to LSQR bidiagonalization:

$$A\,V_k = U_{k+1}\,B_k, \qquad U_{k+1} \text{ and } V_k \text{ have orthonormal columns.}$$

The CGLS/LSQR solution is

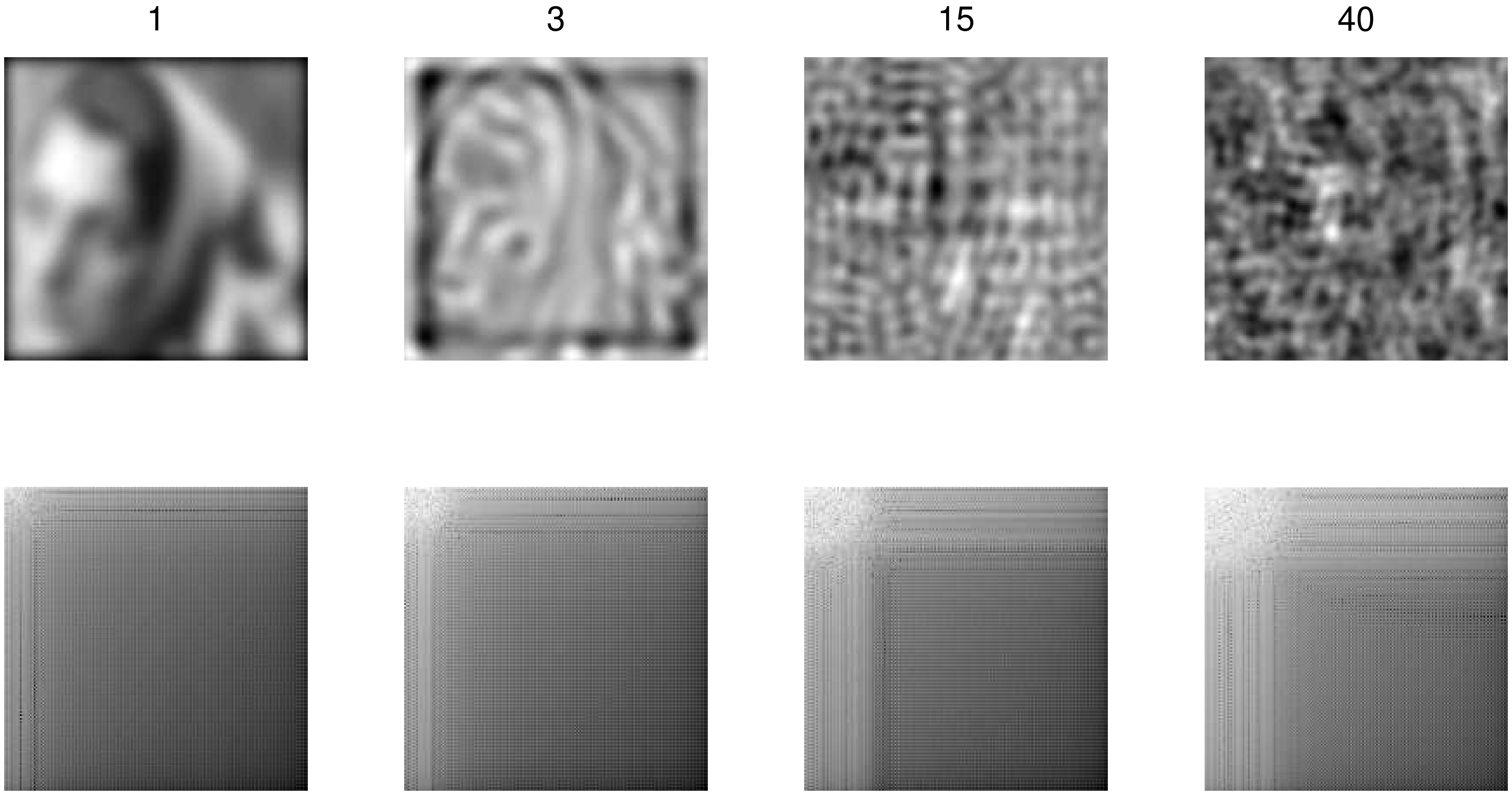
$$\boxed{x^{(k)} = V_k\,\xi_k} \qquad \text{with} \qquad \xi_k = \mathrm{argmin}_\xi \|B_k\,\xi - \beta\,e_1\|_2.$$

Introducing the SVD, $B_k = P_k \Sigma_k Q_k^T$, we can also write

$$\boxed{x^{(k)} = (V_k\,Q_k)\,\Sigma_k^{-1}(\beta\,P_k^T\,e_1)}$$

There is useful information in the columns of $V_k$ and $V_k\,Q_k$.

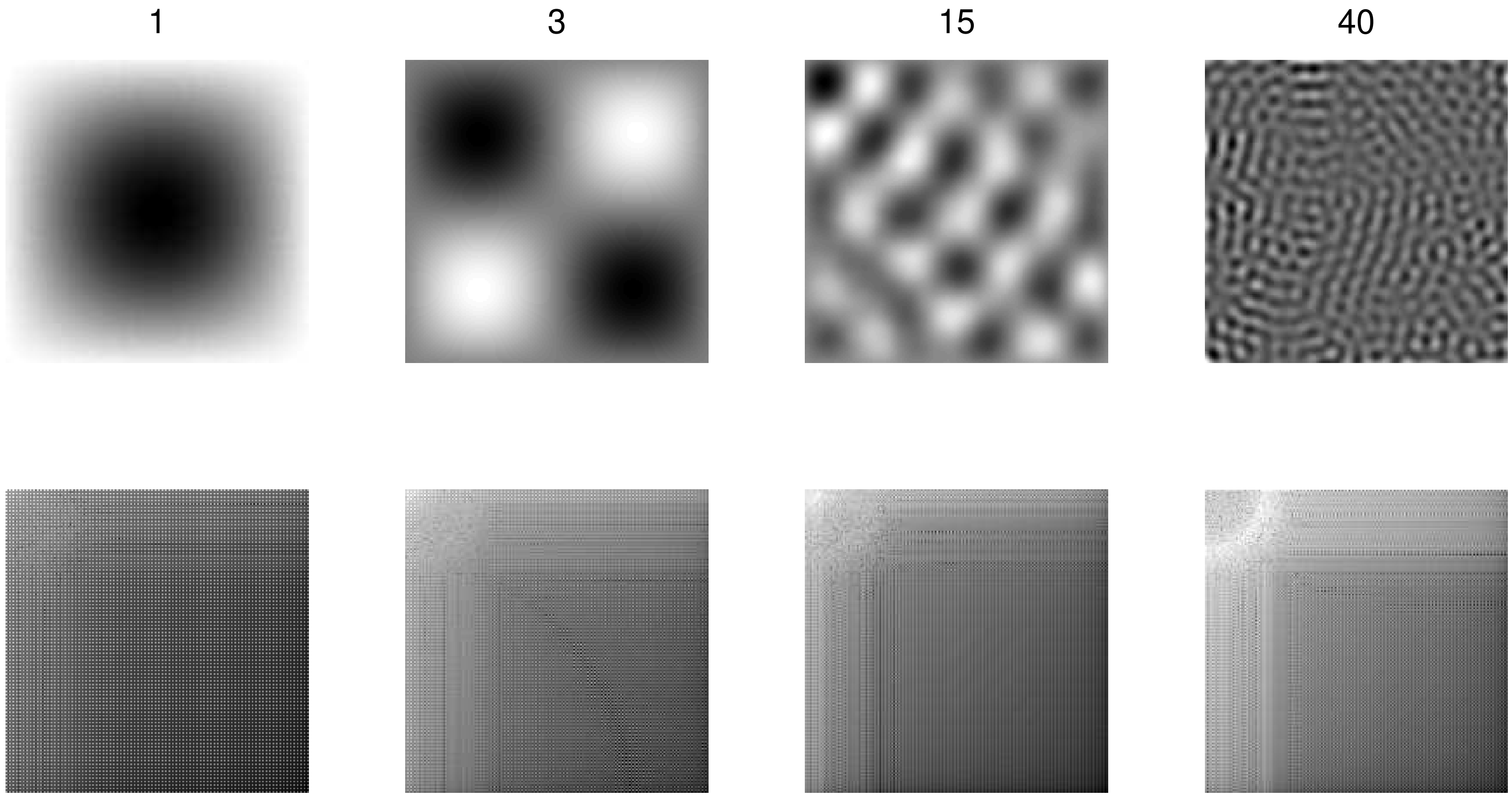
## CGLS Krylov Vectors (Symm. $A$) – Columns of $V_k$




More spectral frequencies are included as $k$ increases.

No high frequencies are present.

## Alt. CGLS Krylov Vectors (Symm. $A$) – Columns of $V_k\,Q_k$




These vectors first resemble spectral bases, but later some "freckles" = band-pass filtered structures begin to appear.

## A Closer Look at the Krylov Basis – GMRES

Again the "power vectors" $A^{i-1}b$ of the GMRES Krylov subspace are just more and more blurred versions of $B$.

Recall that GMRES produces a "Hessenbergization:"

$$A\,W_k = W_{k+1}\,H_k, \qquad W_{k+1} \text{ has orthonormal columns.}$$

The GMRES solution is

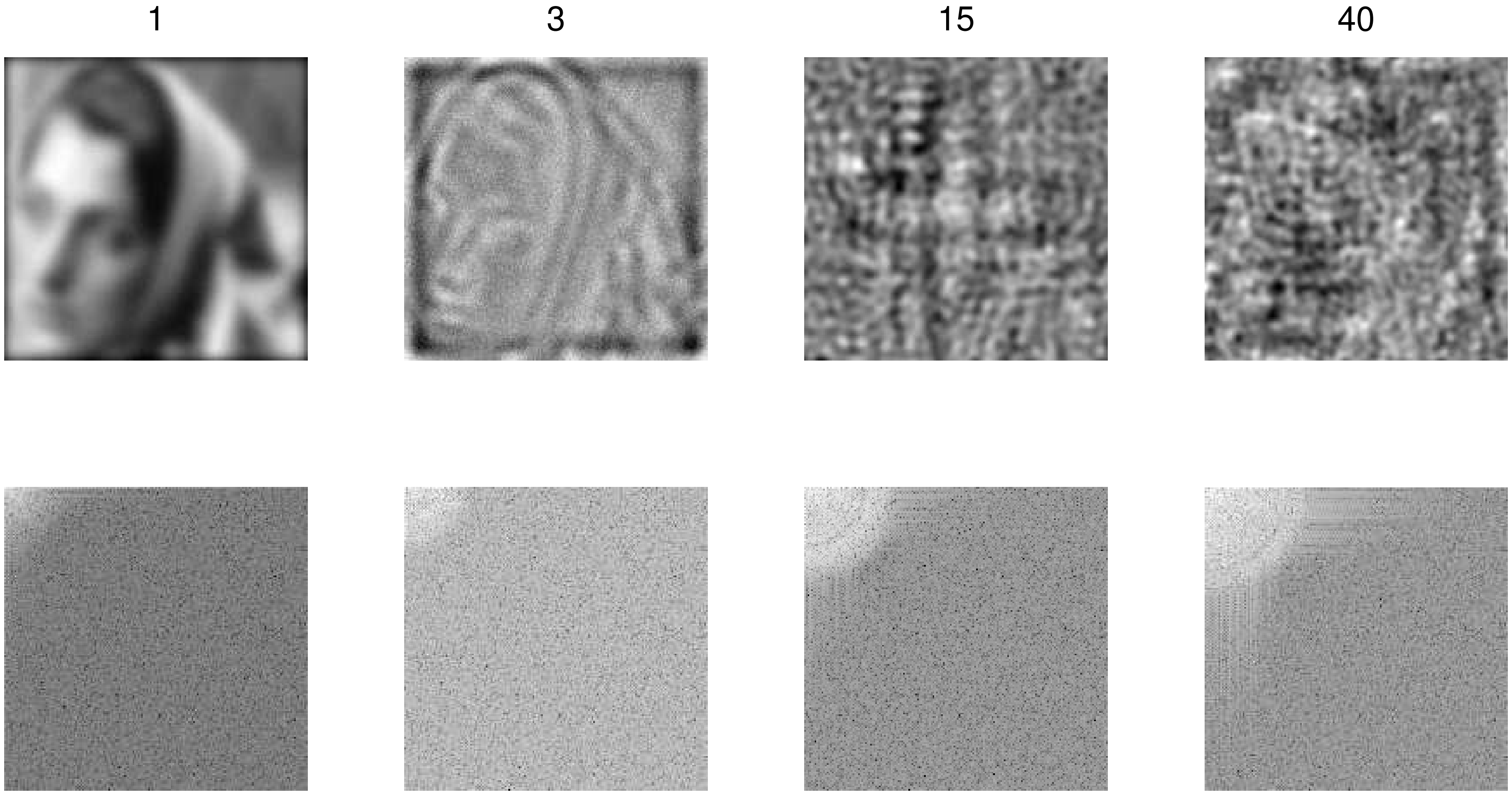
$$\boxed{x^{(k)} = V_k\,\xi_k} \qquad \text{with} \qquad \xi_k = \text{argmin}_\xi \|H_k\,\xi - \beta\,e_1\|_2.$$

Introducing the SVD $H_k = P_k \Sigma_k Q_k^T$ we can also write

$$\boxed{x^{(k)} = (W_k\,Q_k)\,\Sigma_k^{-1}(\beta\,P_k^T e_1)}$$

There is useful information in the columns of $W_k$ and $W_k\,Q_k$.

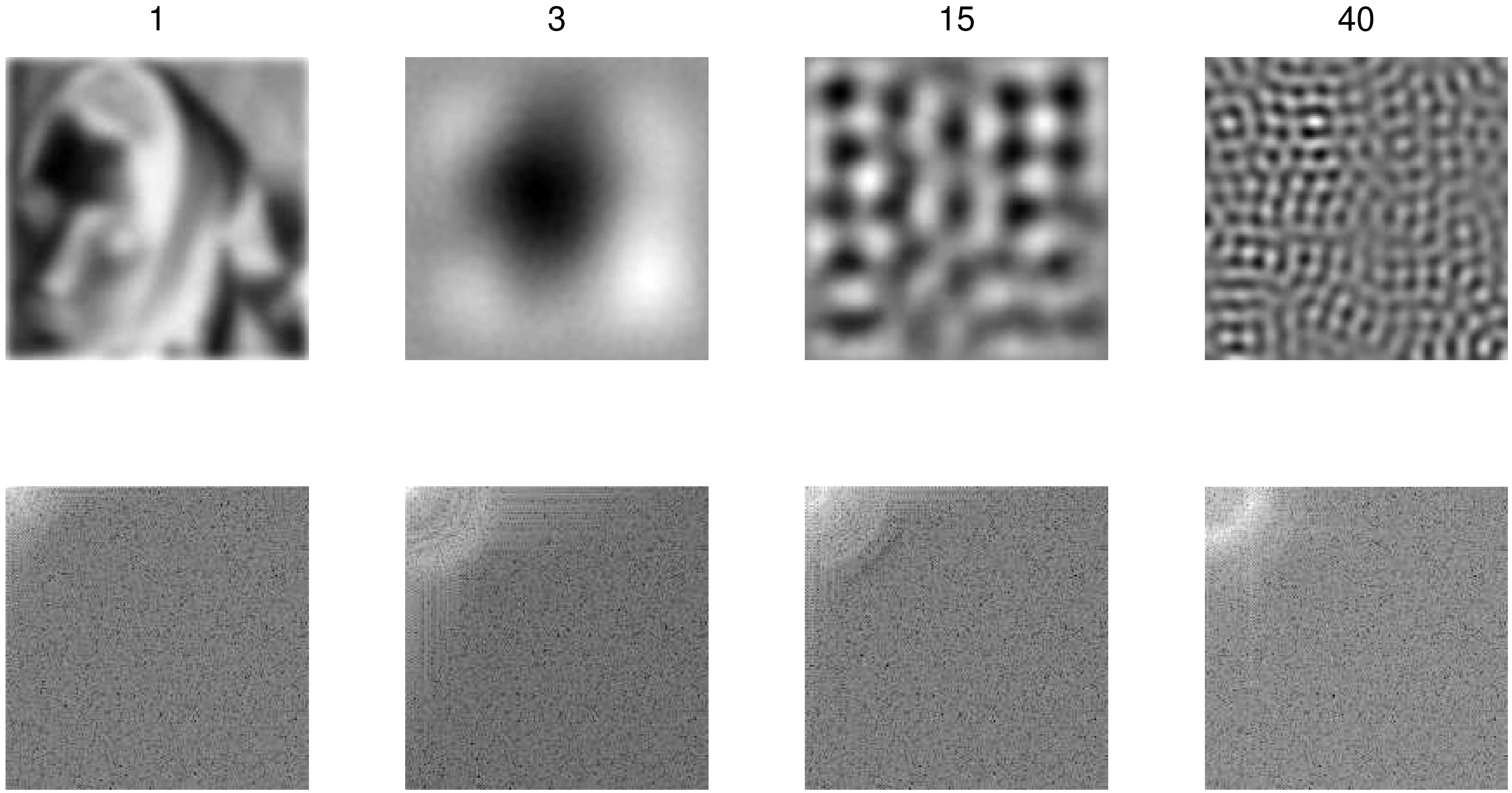
## The GMRES Krylov Vectors – Columns of $W_k$




More spectral frequencies are included as $k$ increases.

White noise is always present, due to the vector $b$ in the basis!

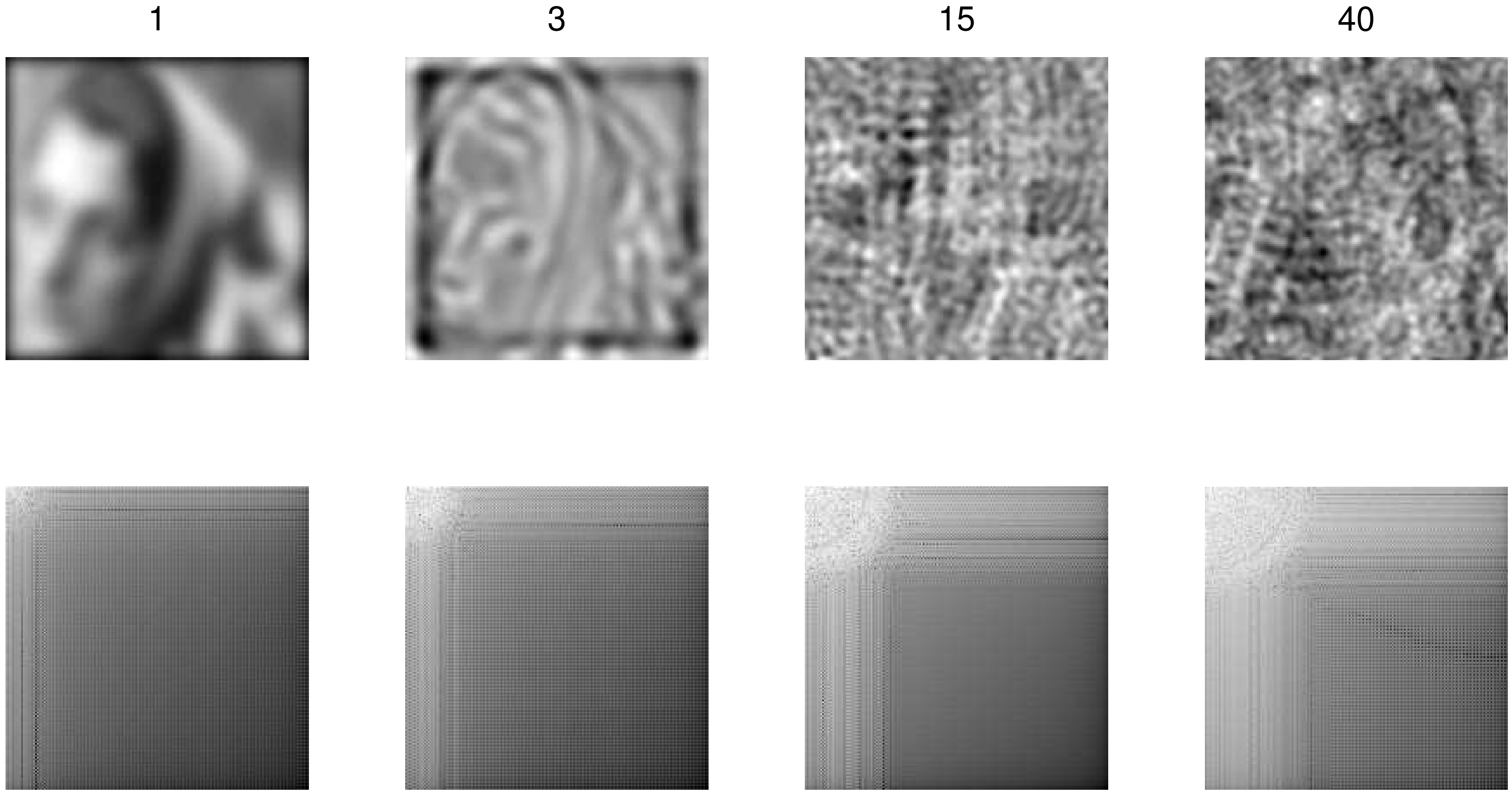
## Alt. GMRES Krylov Vectors – Columns of $W_k\,Q_k$




These vectors do not resemble spectral bases.

The "freckles" = band-pass filtered structures are still present.

## The RRGMRES Krylov Vectors – Columns of $W_k$




Similar to GMRES – but the white noise is gone (thanks to $A\,b$)!

The behavior of RRGMRES resembles that of CGLS.

## How the Noise Enters the Picture

A very common assumption is that the noise in the data is *additive:*

$$b = b^{\text{exact}} + e.$$

This assumption lets us separate the filtered "signal" and "noise" components in the reconstructions.

**LSQR:**

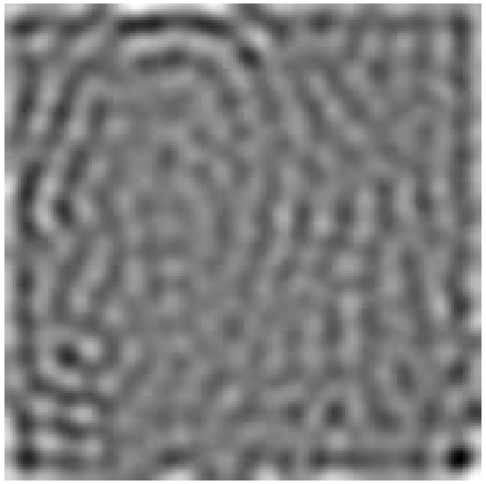
$$x^{(k)} = \mathcal{P}_k(A^T A)\, A^T b = \underbrace{\mathcal{P}_k(A^T A)\, A^T b^{\text{exact}}}_{\text{“signal”}} + \underbrace{\mathcal{P}_k(A^T A)\, A^T e}_{\text{“noise”}}$$

**GMRES:**

$$\hat{x}^{(k)} = \widehat{\mathcal{P}}_k(A)\, b = \underbrace{\widehat{\mathcal{P}}_k(A)\, b^{\text{exact}}}_{\text{“signal”}} + \underbrace{\widehat{\mathcal{P}}_k(A)\, e}_{\text{“noise”}}$$

Here $\mathcal{P}_k$ and $\widehat{\mathcal{P}}_k$ are polynomials of degree $\leq k$ that depend on $b$.

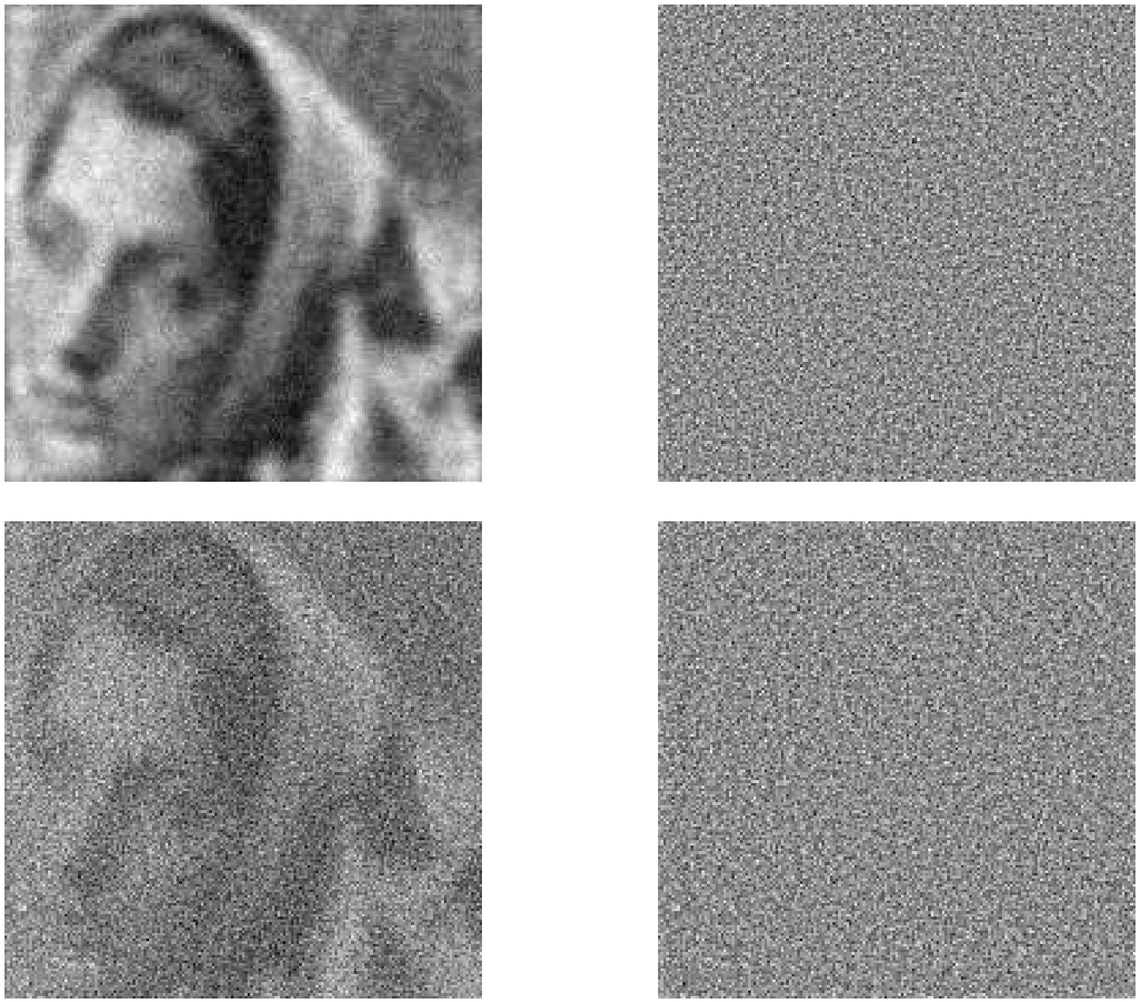
## Signal and Noise Parts in CGLS Iterates ($k = 10$)



Symm. $A$

Nonsym. $A$

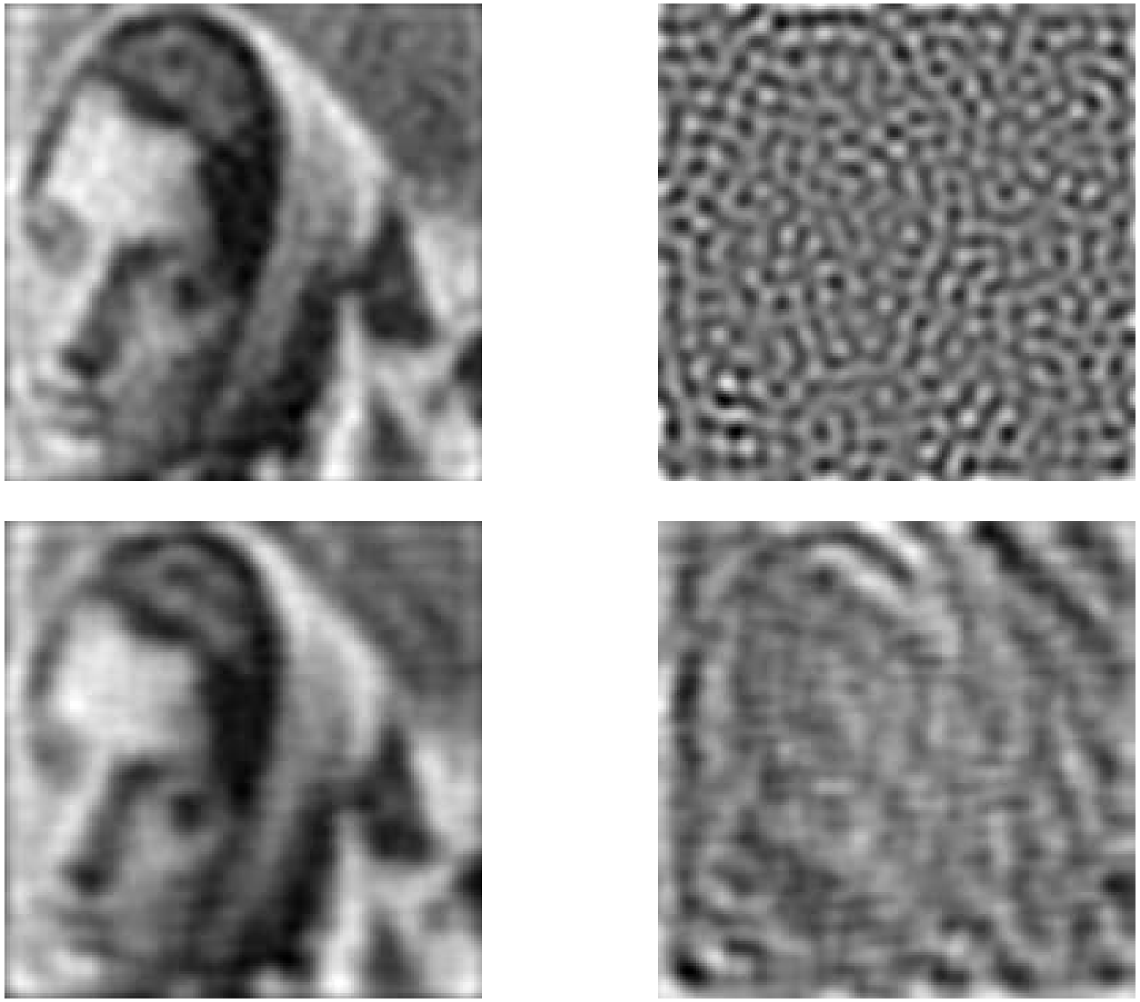
## Signal and Noise Parts in GMRES Iterates ($k = 10$)



Symm. $A$

Nonsym. $A$

## Signal and Noise Parts in RRGMRES Iterates ($k = 10$)



Symm. $A$

Nonsym. $A$

## Observations on the Noise Propagation

- **CGLS:**
  - Signal: getting sharper than the blurred data $B$.
  - Noise: medium-frequency "freckles" and "ringing" that follow the contours in the image.
- **GMRES:**
  - Both signal and noise have a significant white-noise component due to the noise vector in the Krylov subspace.
- **RRGMRES:**
  - Quite similar to CGLS.

We conclude that it is highly advisable to use RRGMRES (and MR-II) instead of GMRES (and MINRES).

There is no clear winner among CGLS and RRGMRES here.

## Lower Noise Level

GMRES has been reported – by others – to be superior to CGLS.

For a problem with very *low noise level*, $\|e\|_2/\|b^{\text{exact}}\|_2 \approx 10^{-4}$, we again separate the signal and noise components → see next page.

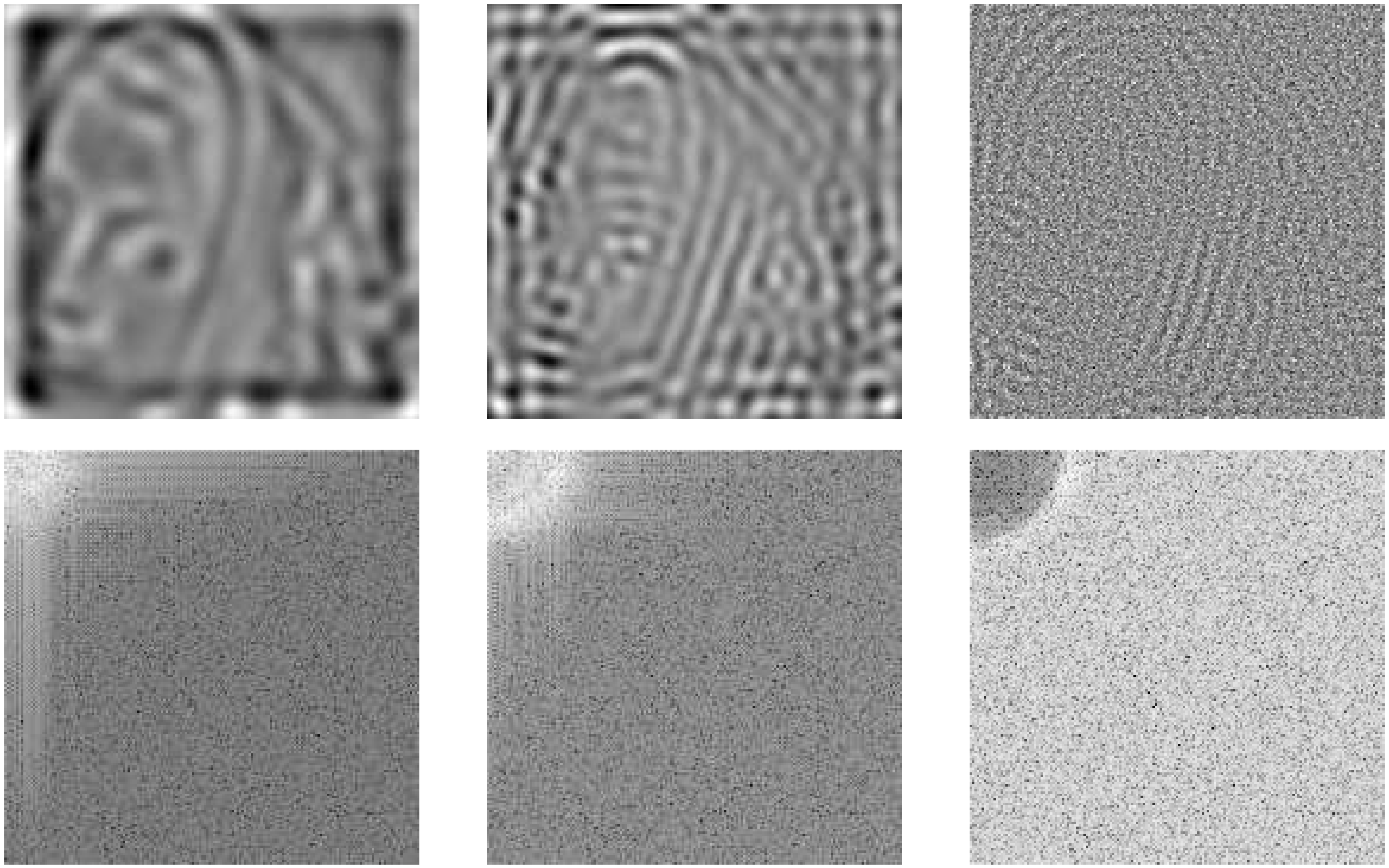
- For $k \leq 25$ we have $\|\text{“noise”}\|_2 \ll \|\text{“signal”}\|_2$.
- Random “freckles” do not appear to the same extent.
- Some “ringing” is visible around the structures in the image
- As $k$ increases the error component becomes much more high-frequent $\sim$ *high-pass filter*.

Conclusions for very low noise levels only:

- GMRES’s high-frequency noise component is less disturbing to the eye – and harder to spot – than LSQR’s “freckles.”
- But the noise component in both methods is *of the same size.*

# DCT Study of GMRES Low-Noise Component

$k = 4$ $\qquad$ $k = 10$ $\qquad$ $k = 25$



## Conclusions

1. Singular vectors and Krylov subspace vectors $\sim$ spectral basis.
2. CGLS and RRGMRES/MR-II achieve regularization via projection on the Krylov subspace.
3. Don't use GMRES/MINRES (noise component in basis).
4. Freckles arise in both CGLS and RRGMRES, no clear winner.
5. Stopping criterion – an open problem (could be based on the projected problem (of small dimensions).
6. Next stage: nonnegativity constraints.
7. Next next stage: use DCT as basis for projection method?