Algebraic Multilevel Methods

with Aggregations

Radim Blaheta
Institute of Geonics AS CR, Ostrava, Czech Republic
www.ugn.cas.cz

Computational Methods with Applications '07,
Harrachov, August 19-25, 2007

Aims of the talk

to show some features of algebraic multilevel methods with aggregations, compare results for conforming/nonconforming FEM

Contents:

- **geometric FE hierarchy and its application: ML preconditioners and hierarchical error estimators**
- **algebraic constructions of hierarchy - aggregation methods for nonconforming and conforming FE's**
- **to describe the hierarchy by local h-refinement**

Solved problems - selfadjoint PDE's

BVP in $\Omega \Rightarrow$ find : $u \in V(\Omega)$: $a(u,v) = b(v) \quad \forall v \in V(\Omega)$,

$$a(u,v) = \int_\Omega \langle D\nabla u, \nabla v\rangle \, dx, \quad a(u,v) = \int_\Omega \langle C\varepsilon(u), \varepsilon(v)\rangle \, dx,$$

D ... matrix of diffusion coefficients *physical anisotropy*,

C ... matrix of elasticity coefficients, *physical anisotropy*,

D, C ... SPD, piecewise constant ,

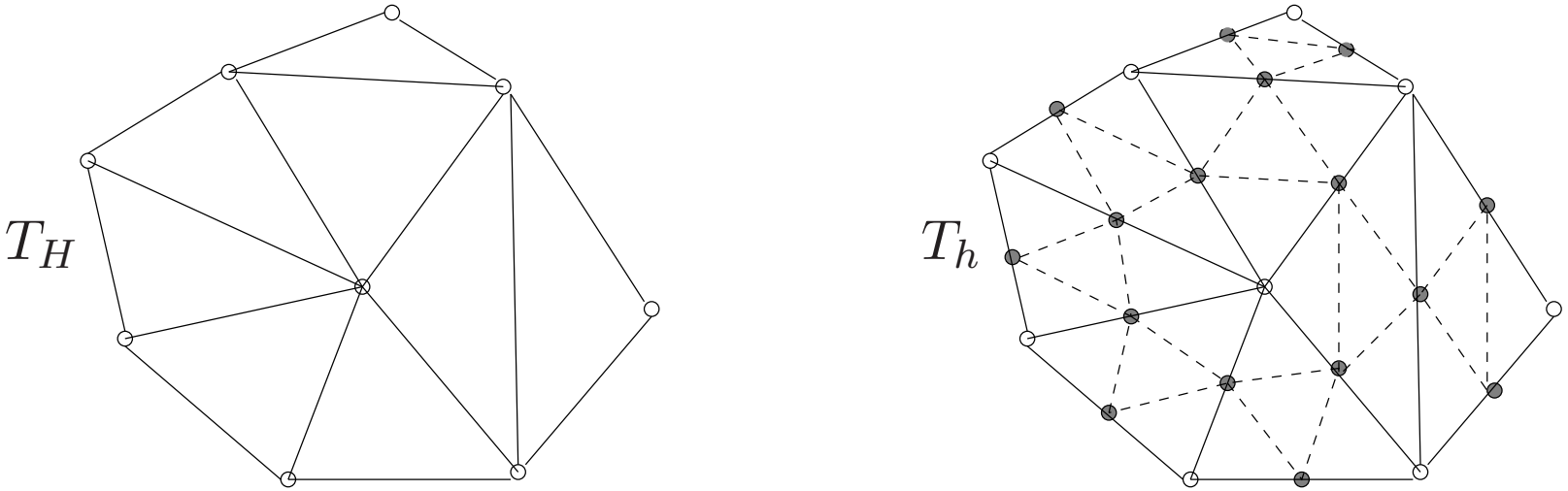
$$V = \left\{ v \in \left(H^1(\Omega)\right)^{1,(d)}, v(x) = 0 \text{ on } \Gamma_0 \right\}$$

FEM: linear triangular/tetrahedral FE. *Numerical anisotropy.*

$V_h = \text{span}\{\phi_i\}, \quad \phi_i \subset V(\Omega), \ \phi_i(x_j) = \delta_{ij}$

$V_h = \text{span}\{\Phi_{ik}\}, \ \Phi_{ik} \subset V(\Omega), \ \Phi_{ik}(x_j) = \delta_{ij} \, (\delta_{1k}, \delta_{2k})^T$

Geometric FE hierarchy $V_h = V_H \oplus V_H^+$

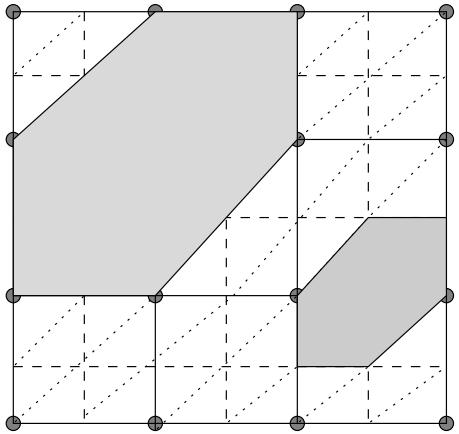


FE grid hierarchy -> FE spaces: $V_0 \subset V_1 \subset \ldots \subset V_m \subset H^1(\Omega)$ **.**

Usefulness:

- increasing accuracy of FE approximation
- building of optimal iterative solvers and preconditioners
- hierarchical error estimators

Nodal and hierarchical bases (NB, HB)



- **coarse grid $\mathcal{T}_H$, nodes $\mathcal{N}_H$, nodal basis $\left\{\phi_i^H\right\} \subset V_H$**
- **fine grid $\mathcal{T}_h$, nodes $\mathcal{N}_h$, nodal basis $\left\{\phi_i^h\right\} \subset V_h$**

- **decomposition $\mathcal{N}_h = \mathcal{N}_H \cup \mathcal{N}_H^+ \Rightarrow$ hierarchical basis $\left\{\bar{\phi}_i^h\right\} \subset V_h$:**

$$\bar{\phi}_i^h = \begin{cases} \phi_i^h & \text{if } x_i \in \mathcal{N}_h^+ \\ \phi_i^H & \text{if } x_i \in \mathcal{N}_H \end{cases} \Rightarrow \text{space decomposition } V_h = V_H \oplus V_H^+$$

- elasticity: $\left\{\Phi_{ik}^H\right\}$, $\left\{\Phi_{ik}^h\right\}$, $\Phi_{ik}(x_j) = \delta_{ij}\,(\delta_{1k},\, \delta_{2k})^T$

Hierarchical decomposition, NB and HB matrices

$$V_h = V_H \oplus V_H^+, \quad V_H = \text{span}\left\{\phi_i^H,\ x_i \in \mathcal{N}_H\right\},\ \ V_H^+ = \text{span}\left\{\phi_i^h,\ x_i \in \mathcal{N}_H^+\right\}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{matrix} \phi_i^h,\ x_i \in \mathcal{N}_H^+ \\ \phi_i^{\mathbf{h}},\ x_i \in \mathcal{N}_H \end{matrix} \qquad \bar{A} = \begin{bmatrix} A_{11} & \bar{A}_{12} \\ \bar{A}_{12}^T & A_H \end{bmatrix} \begin{matrix} \phi_i^h,\ x_i \in \mathcal{N}_H^+ \\ \phi_i^{\mathbf{H}},\ x_i \in \mathcal{N}_H \end{matrix}$$

A both diagonal blocks with local information. $\bar{A}$ second block with global information.

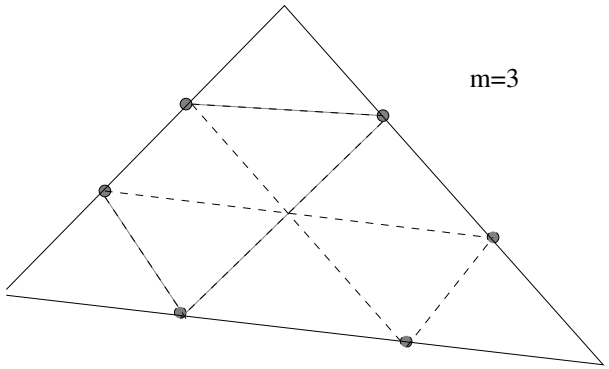
Strengthened C.B.S. inequality: there is $0 \le \gamma < 1$ **:**

$$a(u,\, v) \le \gamma \sqrt{a(u,\, u)}\, \sqrt{a(v,\, v)} \quad u \in V_H,\ v \in V_H^+,$$

$$\left\langle \bar{A}_{12}\, \bar{v}_2,\, \bar{v}_1 \right\rangle \ \ \le \ \ \gamma \sqrt{\left\langle \bar{A}_{11}\, \bar{v}_1,\, \bar{v}_1 \right\rangle}\, \sqrt{\left\langle \bar{A}_{22}\, \bar{v}_2,\, \bar{v}_2 \right\rangle}.$$

(importance of off-diagonal terms, angle of subspaces, etc.)

CBS - universal estimates for P_1 elements



How γ depends on ?

- meshsize h
- refinement multiplicity m
- coefficient anisotropy
- element shape
- coefficient jumps

- m fold refinement !
- anisotropic Laplacian
- general elasticity $c_{i,j,k,l}$!
- arbitrary element shape

$$\left.\begin{array}{l} \gamma \leq \sqrt{\frac{m^2-1}{m^2}} \text{ in 2D,} \\ \\ \gamma \leq \sqrt{1-\frac{2}{m^4+m^2}} \ i\text{n 3D} \end{array}\right\}$$

Axelsson, Blaheta Appl. Math. 2004, *Blaheta* NLAA 2003

Two-level multiplicative preconditioner for A_{NB}

$$A = A^{(k+1)} = \begin{bmatrix} I & \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & \mathbf{S} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$B_M = B_M^{(k+1)} = \begin{bmatrix} I & \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & \\ & \mathbf{\bar{A}_{22}} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ & I \end{bmatrix}$$

Implementation in 3 steps, solving with A_{11}, A_{22}, A_{11}.

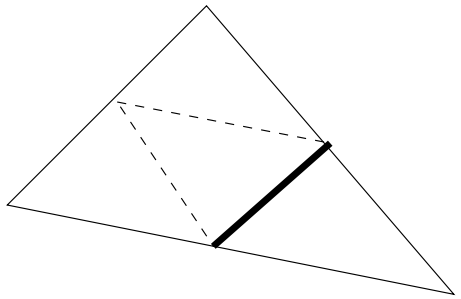
Efficiency of preconditioning:

$$B_M^{-1}A = \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & \\ & \mathbf{\bar{A}_{22}^{-1}S} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$\sigma(\mathbf{B_M^{-1}A}) \subset \sigma(I) \cup \sigma(\bar{A}_{22}^{-1}S) = \sigma(I) \cup \sigma(\bar{A}_{22}^{-1}\bar{S}) \subset \langle \mathbf{1} - \gamma^{\mathbf{2}}, \mathbf{1} \rangle$$

Optimal multi-level preconditioners

- Solve A_{11}:
  - $\sim$ diag(A_{11}) for isotropic Laplacian optimal preconditioner
  - $\sim$ block3diag(A_{11}) robust for anisotropic Laplacian in 2D
- Solve A_{22}:
  - few Chebyshev or GCG iterations with the same ML preconditioner



Axelsson, Padiy 1999

If everything is well balanced, we get optimal method !

Axelsson 1981, Axelsson, Gustafsson 1983, Axelsson, Vassilevski 1989, 1990, ...

Hierarchical Aposteriori Error Estimation

- $V_h = V_H \oplus V_H^+ \longrightarrow u_H,\ u_h,$ **estimate** $\|u - u_H\|_E$???
- an **expensive estimate** uses $w = u_h - u_H$

$$(1+\beta)^{-1}\|w\|_E \le \|u - u_H\|_E \le (1-\beta)^{-1}\|w\|_E\,,$$

$$\|w\|_E \le \|u - u_H\|_E \le (1-\beta^2)^{-1/2}\|w\|_E\,,$$

  if we have the saturation property: $\|u - u_h\|_E \le \beta\,\|u - u_H\|_E\,, \quad \beta < 1$

- a **cheaper estimate** uses $\bar{w}$:
  $\bar{w} \in V_H^+,\ a(u_H + \bar{w},\, v) = b(v)\ \forall\, v \in V_H^+,$ i.e. $a(\bar{w},\, v) = a(u - u_H,\, v)$

$$\|\bar{w}\|_E \le \|u - u_H\|_E \le (1-\beta^2)^{-1/2}(1-\gamma^2)^{-1/2}\|\bar{w}\|_E$$

- **algebraically**, $\bar{w}$ is computed from $A_{11}\bar{w} = b_1 - A_{12}u_H$ and A_{11} can be further approximated by a local operator.

  Bank, Smith - SINUM 1995, Bank - Acta Numerica 1996, . . .

Hierarchical EE with $\widetilde{A}_{11}$

- we compute

$$A_H u_H = \bar{A}_{22} u_2 = b_2$$
$$\widetilde{A}_{11} u_1 = b_1 - A_{12} u_2$$

- then we are able to check the discretization error $\|u - u_H\|_E$ by $\|\bar{w}\|_E = \langle A_{11} u_1,\, u_1 \rangle$
- if the error is not small enough, we can *improve the refinement or use a nonuniform refinement*
- an approximation $\widetilde{A}_{11}$ of A_{11} as e.g. Axelsson, Padiy 1999 gives

$$\underline{\mu}\, \|\bar{w}\|_E \le \|u - u_H\|_E \le \bar{\mu}\, (1-\beta^2)^{-1/2} (1-\gamma^2)^{-1/2}\, \|\bar{w}\|_E$$

where $\underline{\mu} = \left(1 - \sqrt{7/15}\right)$ and $\bar{\mu} = \left(1 + \sqrt{7/15}\right)$ independently on the element shape and anisotropy, $\frac{\bar{\mu}}{\underline{\mu}} = 5.31$.

Robustness w.r.t. μ, γ. What about β?

Algebraic Hierarchy

$$V_h \longrightarrow V_A \sim V_H, \quad V_h = V_A \oplus V_D$$

- **Application 1**: nonconforming FE, V_H is not a subspace of V_h , but there is the decomposition $V_h = V_A \oplus V_D$ with V_A connected with macroelements,

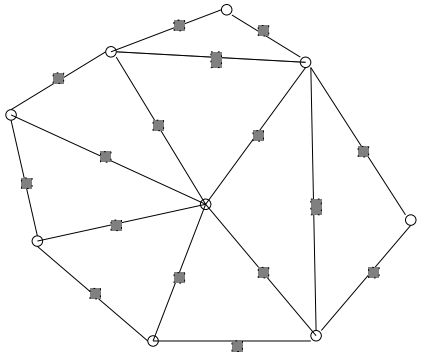
  $$\Omega, \;\; \mathcal{T}_h = \{T\}, \;\; \mathcal{T}_H = \{E\} \quad V_h \leftrightarrow \mathcal{T}_h, \;\; V_A \leftrightarrow \mathcal{T}_H \leftrightarrow V_H$$

- **Application 2**: aggregation, aglomeration methods - again V_A connected with macroelements.

$$\bar{A} = \begin{bmatrix} A_{11} & \bar{A}_{12} \\ \bar{A}_{12}^T & A_{22} \end{bmatrix} \begin{matrix} D \\ A \end{matrix}$$

Nonconforming (Crouzeix-Raviart) FE

BVP: find $u \in V(\Omega)$: $\quad a(u,v) = b(v) \quad \forall v \in V(\Omega)$



$\Omega \rightarrow$ triangulation $\mathcal{T}_h$

$\mathcal{T}_h \rightarrow M_h$ - set of mid points

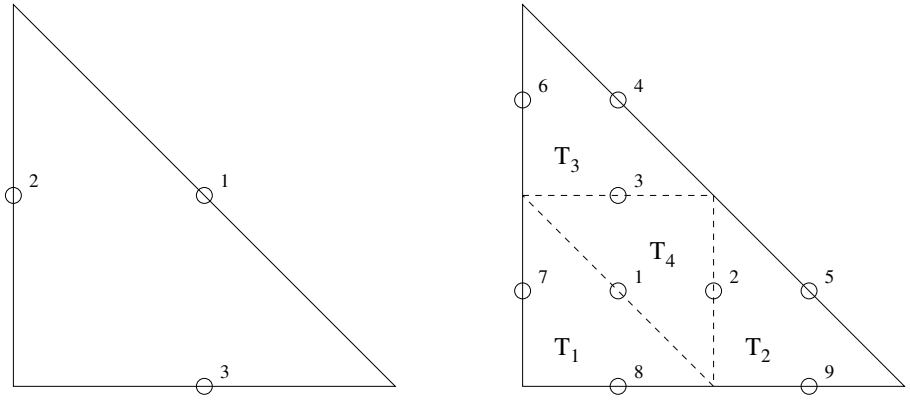
$[v](x)$ - jump of v in x on an edge of T

- $U_h^{CR} = \{v \in L_2(\Omega) : \ v|_T \in P_1 \ \forall T \in \mathcal{T}_h, \ [v](x) = 0 \ \forall x \in M_h\}$
- $V_h = \left\{v \in \widetilde{U}_h : \ v(x) = 0 \ \forall x \in M_h \cap \Gamma_0\right\}$
- $a_h(u,v) = \sum_{T \in \mathcal{T}_h} \int_T \langle D\nabla u, \nabla v\rangle \, dx$

find $u_h \in V_h(\Omega)$: $\quad a_h(u_h, v_h) = b(v_h) \quad \forall v_h \in V_h(\Omega)$

- $u \in H^2(\Omega) \Rightarrow \|u - u_h\|_h \leq ch\,|u|_2$

An algebraic hierarchy for Crouzier-Raviart FE



$V_H^{CR} \not\subseteq V_h^{CR}$!

basis functions on

macroelement E: $\phi_1 \dots \phi_9$

DA construction: $V_h = V_A \oplus V_D$,

$$V_A(E) = \text{span}\{\widetilde{\phi}_7 \dots \widetilde{\phi}_9\} = \text{span}\{\phi_1 + \phi_4 + \phi_5, ..\} \quad \text{“aggregation”},$$
$$V_D(E) = \text{span}\{\widetilde{\phi}_1 \dots \widetilde{\phi}_6\} = \text{span}\{\phi_1, .., \phi_4 - \phi_5, ..\}$$

NB $\phi_1 \dots \phi_9 \xrightarrow{J}$ **HB** $\widetilde{\phi}_1 \dots \widetilde{\phi}_9 \qquad \bar{A}_h = JAJ^T$

$$\bar{A}_h = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{matrix} V_D \\ V_A \end{matrix} \qquad \rightarrow \mathbf{\bar{A}_{22} = 4A_H}$$

CBS inequality for DA decomposition

$$\gamma_{DA}: \quad \langle \bar{A}_{12} y_2, y_1 \rangle \leq \gamma_{DA} \sqrt{\langle \bar{A}_{11} y_1, y_1 \rangle} \sqrt{\langle \bar{A}_{22} y_2, y_2 \rangle}$$

Theorem (RB, S. Margenov, M. Neytcheva, NLAA 2004):
Let us consider nested triangular grids, with coarse triangles consisting from 4 smaller ones. Assume that the coefficient matrix D has no jumps within the coarse triangles. Then

$$\gamma_{DA} = \sqrt{\frac{3}{4}}$$

for any element size and shape and general anisotropic Laplacian.

Preconditioners to $\bar{A}_{11}$ - EBE elimination of inner DOF's and then e.g. Axelsson, Padiy construction

Preconditioning and EE via Algebraic Hierarchy

- preconditioning - the same schemes as for geometric hierarchy
- error estimate

$$V_h$$

$$\downarrow$$

$$V_A \sim V_H,\ V_A \oplus V_D$$

$$computation\ u_A,\ w_A$$

$$evaluation\ of\ error\ distribution$$

$$on\ macroelements$$

OK	$not\ OK$
$continue\ with$	$refine$
$AML\ method$	$new\ V_h$

Hierachical error estimate for CR FEM

- $V_h = V_A \oplus V_D$
- $u_A: \quad \bar{A}_{22} u_A = b_2$
- $u_H: \quad A_H u_H = b_H$
- $e: \quad \bar{A}_{11} e = b_1 - A_{12} u_A$ or $\tilde{e}: \quad \tilde{A}_{11} \tilde{e} := b_1 - A_{12} u_A$

1. $(1+\beta)^{-1} \|u_H - u_h\|_h \leq \|u - u_H\|_H \leq (1-\beta)^{-1} \|u_H - u_h\|_h$
2. $\|u_H - u_h\|_h \sim \|u_A - u_h\|_h$ **only $\leq$!**
3. $\| e \|_h \leq \|u_A - u_h\|_h \leq (1-\gamma^2)^{-1/2} \| e \|_h$

   or $\underline{\mu} \| \tilde{e} \|_h \leq \|u_A - u_h\|_h \leq \bar{\mu}(1-\gamma^2)^{-1/2} \| \tilde{e} \|_h$

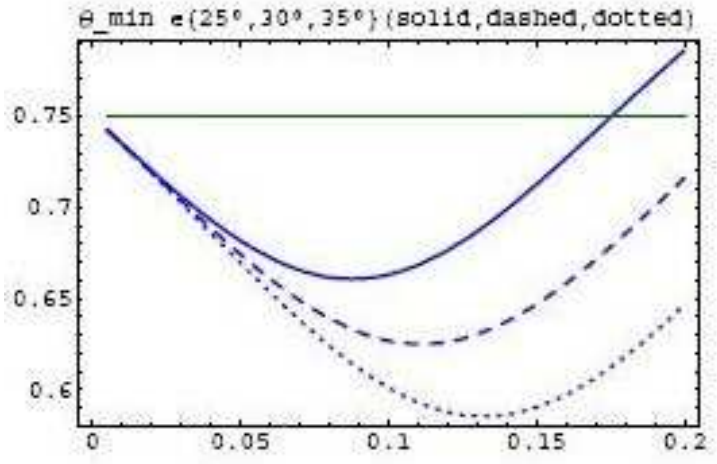
Smoothed (generalized) aggregations

GDA construction (Kraus, Margenov, Synka 2006-7):

$V_h = V_A \oplus V_D$,

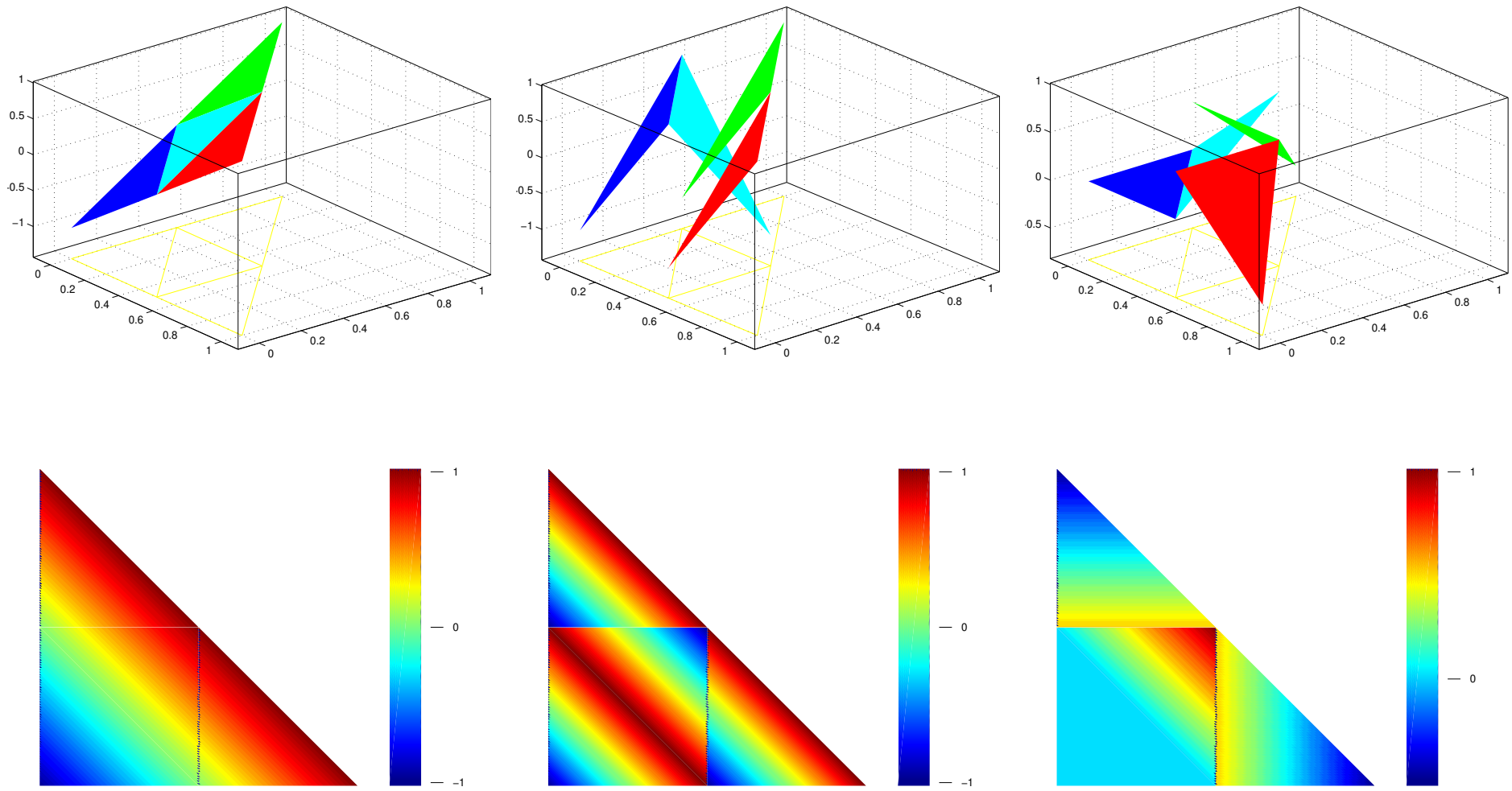
$$V_A(E) = \text{span}\{\widetilde{\phi}_7 \dots \widetilde{\phi}_9\} = \text{span}\{\phi_1^C + \phi_4 + \phi_5, ..\} \quad ,$$
$$V_D(E) = \text{span}\{\widetilde{\phi}_1 \dots \widetilde{\phi}_6\} = \text{span}\{\phi_1, .., \phi_4 - \phi_5, ..\}$$

- GDA $\phi_i^C = \sum_k c_{ik}\phi_k, \quad C = (c_{ik}) = I + 2\mu(\mathbf{1} - 3I)$
- FR $C = -A_{11}^{-1}\bar{A}_{13}$ - represent. A in inner DOF, differ., aggreg.



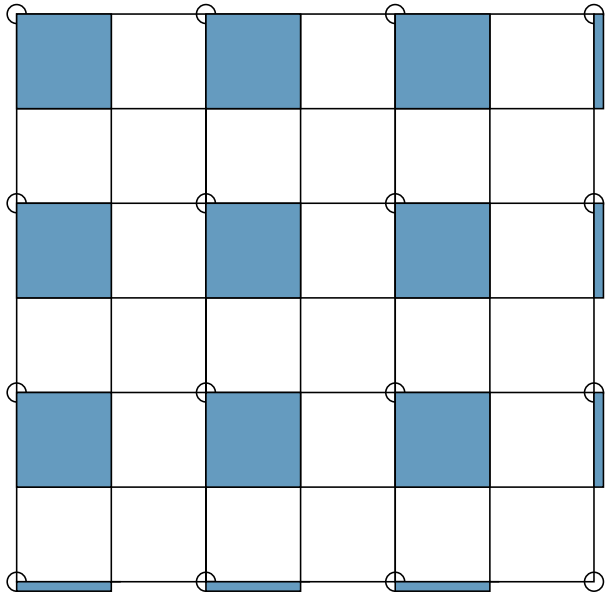
GDA
Margenov, Synka
NM&A '06

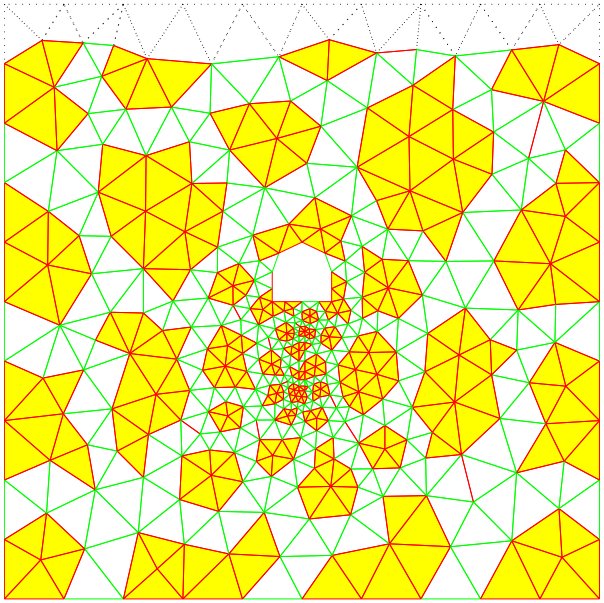
Approximation properties



DA and GDA decomposition - local basis function

Algebraic hierarchy by aggregation





Aggregation = selecting C-nodes (o) and F-nodes (remaining). Split F-nodes to groups associated with C-nodes. Regular/irregular aggreg.

$$\phi_i^{HB} = \sum_k J_{ik}\,\phi_i^h, \qquad J = (J_{ij}) = \begin{bmatrix} I_1 & 0 \\ I_{21} & I_2 \end{bmatrix},$$

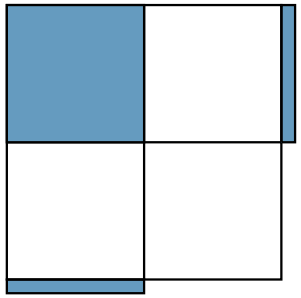
I_{21} Boolean, one unity per column

$$\Rightarrow \quad V_h = V_A \oplus V_D$$

[Blaheta 1986, 1989, Braess 1994, Notay 2002, 2006]

CBS for algebraic hierarchy by aggregation

$V_h = V_a \oplus V_a^+$, $a(u,v) = \sum_E a_E(u_E, v_E)$, E macroelements:



$$D = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 & \\ & 10^k \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

γ :

k=	0	1	2	3	4	5	6
0°	.8966	.9080	.9123	.9128	.9129	.9129	.9129
15°	.8966	.9210	.9710	.9961	.9996	1.000	1.000
30°	.8966	.9391	.9879	.9997	.9999	1.000	1.000
45°	.8966	.9453	.9907	.9990	.9999	1.000	1.000

Hierachical error estimate for aggregated FE

- $V_h = V_A \oplus V_D$
- $u_A: \quad \bar{A}_{22} u_A = b_2$
- $u_H: \quad A_H u_H = b_H$
- $e: \quad \bar{A}_{11} e = b_1 - A_{12} u_A$ or $\tilde{e}: \quad \tilde{A}_{11} \tilde{e} := b_1 - A_{12} u_A$

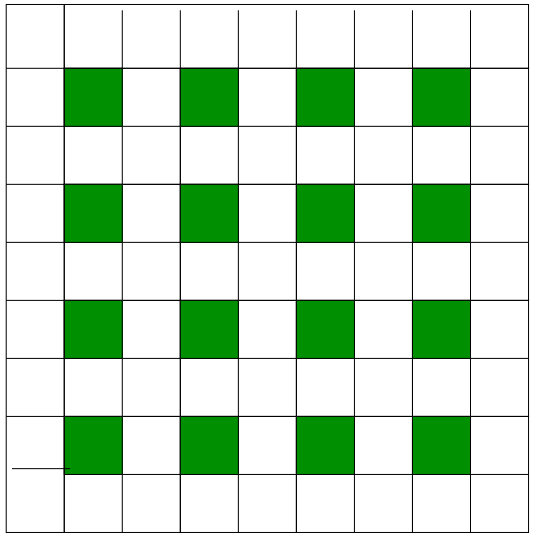
1. $(1+\beta)^{-1} \|u_H - u_h\|_h \leq \|u - u_H\|_H \leq (1-\beta)^{-1} \|u_H - u_h\|_h$
2. $\|u_H - u_h\|_h \sim \|u_A - u_h\|_h$ **only $\leq$!** $\|u_A - u_h\|_h \geq \|u_h\|_{h,\Omega_a}$
3. $\| e \|_h \leq \|u_A - u_h\|_h \leq (1-\gamma^2)^{-1/2} \| e \|_h$

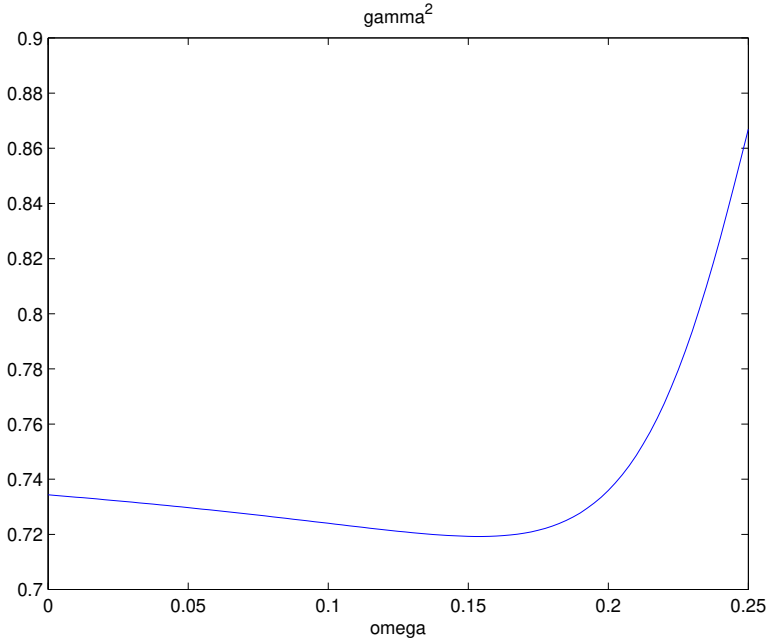
   or $\underline{\mu} \| \tilde{e} \|_h \leq \|u_A - u_h\|_h \leq \bar{\mu}(1-\gamma^2)^{-1/2} \| \tilde{e} \|_h$

Smoothed aggregations

Vaněk 1992, Brezina, Mandel, Vaněk 1996, 2000

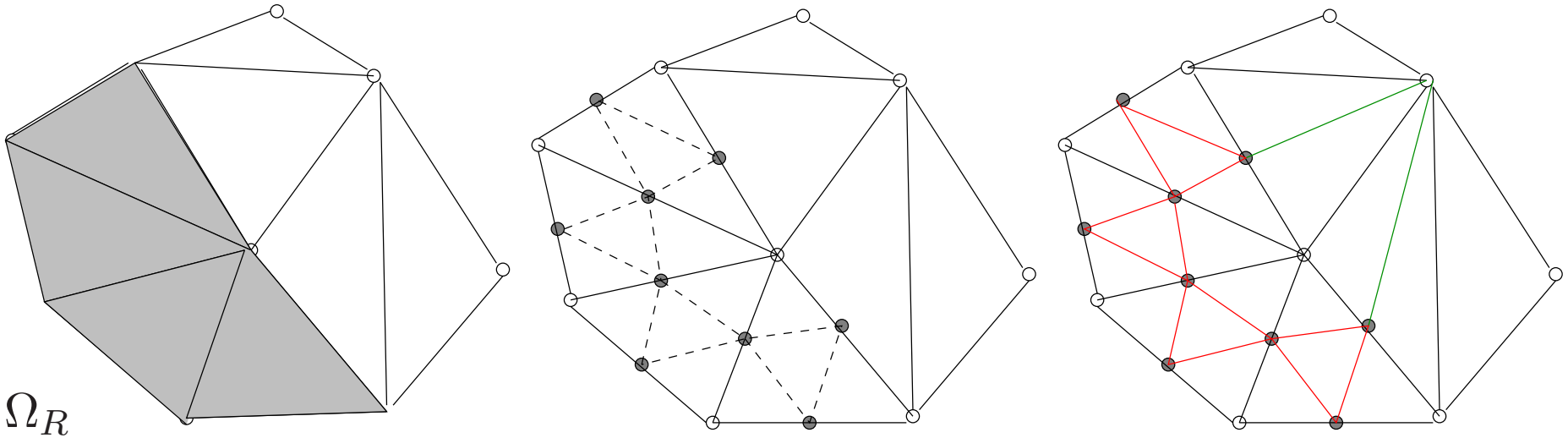
- $\Delta u = f$ in Ω
- $u = 0$ on $\partial\Omega$
- 10x10 grid, regular 2x2 aggreg
- $I_C = (1 - \omega A)\, I_A$, I_F
- $A_{CC} = I_C^T A I_C$, $A_{CF} = I_C^T A I_F$,
- $A_{FF} = I_F^T A I_F$





$\gamma = 0.8569$ for $\omega = 0$ $\quad \gamma = 0.8481$ for $\omega = 0.155$ $\quad Sv = \lambda \bar{A}_{CC} v,\ \lambda_{min} = 1 - \gamma^2$

FE hierarchy by local refinement



- refinement with hanging nodes $\mathcal{N}_H^+(\partial\Omega_R)$

  $\mathcal{N}_R = \mathcal{N}_H(\Omega) \cup \mathcal{N}_H^+(\Omega_R) \cup \mathcal{N}_H^+(\partial\Omega_R), \quad V_R = V_H \oplus V_H^+$

  $V_H = \text{span}\left\{\phi_i^H,\ x_i \in \mathcal{N}_H\right\},\ \ V_H^+ = \text{span}\left\{\phi_i^h,\ x_i \in \mathcal{N}_H^+(\Omega_R)\right\}$

- conforming red-green refinement $V_h = V_H \oplus V_H^+$

  $V_H = \text{span}\left\{\phi_i^H,\ x_i \in \mathcal{N}_H\right\},\ \ V_H^+ = \text{span}\left\{\phi_i^h,\ x_i \in \mathcal{N}_H^+\right\}$

Local refinement with irregular (hanging) nodes

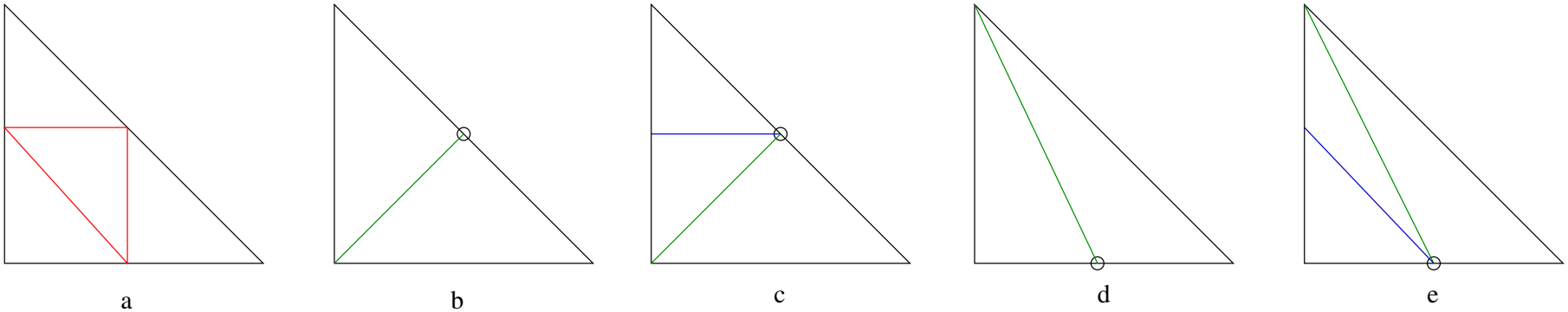
- let $u \in V_H,\ v \in V_H^+$, and E be an macroelement. Then $u\,|_E$ is linear, $v\,|_E$ is piecewise linear, $v = 0$ in vertices of E plus added constrains. Thus, we get the same estimate for γ as for global refinement.
- for implementation, we can use A_H from $\mathcal{T}_H(\Omega)$ and A_R from $\mathcal{T}_h(\Omega_R)$:

$$A_H = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{matrix} \ldots \mathcal{N}_H \cap \bar{\Omega}_R \\ \ldots \mathcal{N}_H \setminus \bar{\Omega}_R \end{matrix} \qquad \bar{A}_R = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} \ldots \mathcal{N}_H^+ \\ \ldots \mathcal{N}_H \cap \bar{\Omega}_R \end{matrix}$$

$$\text{then } \bar{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix} \begin{matrix} \ldots \mathcal{N}_H^+ \\ \ldots \mathcal{N}_H \cap \bar{\Omega}_R \\ \ldots \mathcal{N}_H \setminus \bar{\Omega}_R \end{matrix} \qquad \begin{matrix} \mathcal{N}_H^+ \\ \Big\} \mathcal{N}_H \end{matrix}$$

Summary: hanging nodes local refinement work with hierarchical FE matrix, decomposition is robust wrt. material/shape anisotropy

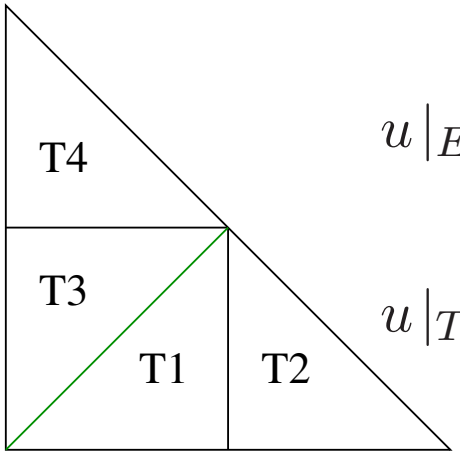
Conforming local refinement



- estimate for the CBS constant for macroelement a)
  $\gamma^2 = \frac{1}{2}$ for orthotropic Laplacian and rectangular triangles¸
  $\gamma^2 = \frac{3}{4}$ for general anisotropy and general triangles
- estimate for the CBS constant for macroelement e) and b) c) d)
  $\gamma^2 = \frac{1}{2}$ for orthotropic Laplacian and rectangular triangles¸
  $\gamma^2 = 1$! for general anisotropy and general triangles

Summary: conforming local refinement provides standard nodal FE matrix, but decomposition deteriorates with material/shape anisotropy

Estimate of CBS constant



$$u|_E \in P_1: \quad \frac{\partial u}{\partial x} = \delta_x, \quad \frac{\partial u}{\partial y} = \delta_y$$

$$u|_{T_k} \in P_1: \quad \frac{\partial v}{\partial x} = d_x^{(k)}, \quad \frac{\partial v}{\partial y} = d_y^{(k)}$$

	T_1	T_2	T_3	T_4
$\frac{\partial v}{\partial x}$	$d_x^{(1)}$	$-d_x^{(1)}$	d_x^3	d_x^3
$\frac{\partial v}{\partial y}$	$d_y^{(1)}$	$d_y^{(1)}$	d_y^3	$-d_y^3$

$$\begin{aligned} a(u,v) &= k_x \delta_x 2 d_x^{(3)} + k_y \delta_y 2 d_y^{(1)} \\ a(v,v) &= k_x [2(d_x^{(1)})^2 + 2(d_x^{(3)})^2] + k_y [2(d_y^{(1)})^2 + 2(d_y^{(3)})^2] \\ a(u,u) &= 4k_x (\delta_x)^2 + 4k_y (\delta_y)^2 \end{aligned}$$

$$a(u,v) \le 2\sqrt{k_x \delta_x^2 + k_y \delta_y^2}\sqrt{k_x (d_x^{(3)})^2 + k_y (d_y^{(1)})^2} \le 2\sqrt{\tfrac{a(u,u)}{4}}\sqrt{\tfrac{a(v,v)}{2}},$$

$$\text{thus } \gamma = \sqrt{\tfrac{1}{2}}$$

Green/blue refinement adds a constraint $\gamma = \sqrt{1/2}$ remains valid.

Summary

- FE hierarchy is useful for both iterative solution methods and error estimation
- robustness with respect to physical and numerical anisotropy
- use of aggregations and smoothing
- analogies between nonconforming/conforming FEM
- related issues - refinement by local aggregation, scaling

  ...

Thank you for your attention.