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Integral combinations of Heavisides

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A sufficiently smooth function of d variables that decays fast enough at infinity can be represented pointwise by an integral combination of Heaviside plane waves (i.e., characteristic functions of closed half-spaces). The weight function in such a representation depends on the derivatives of the represented function. The representation is proved here by elementary techniques with separate arguments for even and odd d, and unifies and extends various results in the literature.

1 Introduction

An integral formula of the form

$$\int_A w(\mathbf{a})\phi(\mathbf{a},\mathbf{x})\,d\mathbf{a}$$

can be metaphorically viewed as a one-hidden-layer neural network with a single linear output unit and a continuum of hidden units. Each hidden unit computes a value of the function ϕ depending on an input vector $\mathbf{x}$ and a parameter vector $\mathbf{a}$. The output function of the network is a weighted integral combination of the hidden unit functions.

Such integral formulas have been used to show that output functions from one-hidden-layer neural networks with suitable ϕ and finitely many units are dense in various function spaces (see, e.g., Funahashi [8], Carroll and Dickinson [4], and Ito [13]). Integral representations have also been used to estimate how accuracy of approximation varies with the number of hidden units (see, e.g., Barron [1], Girosi and Anzellotti [10], and Kůrková, Kainen and Kreinovich [19]). In a neural network with finitely many units the integral is replaced by a Riemann sum. A neural network can even be thought of as a kind of numerical quadrature, a generalization of the midpoint, trapezoid, and Simpson rules for approximating integrals.

In this paper we derive integral formulas corresponding to one-hidden-layer Heaviside networks, extending and unifying results in [8], [4], [13], and [19]. Some of the ideas in this paper appeared in [16]. See also Helgason's monograph [11] on the Radon transform.

An outline of the paper follows. Section 2 reviews neural networks and establishes notation, Section 3 discusses Green's functions and Green's second identity, while Section 4 describes functions of controlled decay and states our main theorem. We consider the 1-dimensional case in Section 5, provide necessary lemmas in Section 6, and prove the main theorem in Section 7. Section 8 has extensions and refinements of our representation and its relation to known results. The paper ends with a brief discussion.

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2 Feedforward neural networks

Feedforward neural networks compute functions determined by the type of units and their interconnections. Each *computational unit* depends on two vector variables (an *input* and a *parameter*), and is given by a function $\phi : \mathcal{R}^p \times \mathcal{R}^d \longrightarrow \mathcal{R}$, where p and d are the dimensions of the parameter and input space respectively and $\mathcal{R}$ denotes the set of real numbers.

One-hidden-layer networks, with hidden units based on a fixed function ϕ and a single linear output unit, yield functions $f : \mathcal{R}^d \longrightarrow \mathcal{R}$ of the form

$$f(\mathbf{x}) = \sum_{i=1}^{n} w_i \phi(\mathbf{a}_i, \mathbf{x}), \tag{2.1}$$

where n is the number of hidden units, and $w_i \in \mathcal{R}$ are the output weights and $\mathbf{a}_i \in \mathcal{R}^p$ the input parameters of the i-th unit for $i = 1, \dots, n$.

A *perceptron* is a computational unit based on a function of the form $\phi((\mathbf{v}, b), \mathbf{x}) = \psi(\mathbf{v} \cdot \mathbf{x} + b)$, where $\psi : \mathcal{R} \longrightarrow \mathcal{R}$ is called the *activation function*, $\mathbf{v} \in \mathcal{R}^d$ is an *input weight* vector, and $b \in \mathcal{R}$ is a *bias*. *Parameter vectors* are pairs $(\mathbf{v}, b) \in \mathcal{R}^{d+1}$, so $p = d + 1$. The activation function often takes values between 0 and 1.

A typical activation function, and the focus of this paper, is the *Heaviside function* ϑ defined by $\vartheta(t) = 0$ for $t < 0$, $\vartheta(t) = 1$ for $t \geq 0$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathcal{R}^d$, and S^{d-1} the unit sphere in $\mathcal{R}^d$. For every $a > 0$, $\vartheta(at) = \vartheta(t)$. So, for $\mathbf{v} \neq 0$, $\vartheta(\mathbf{v} \cdot \mathbf{x} + b) = \vartheta(\mathbf{e} \cdot \mathbf{x} + b')$, where $\mathbf{e} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \in S^{d-1}$ and $b' = \frac{b}{\|\mathbf{v}\|}$.

For $\mathbf{e} \in S^{d-1}$ and $b \in \mathcal{R}$ we denote by $H_{\mathbf{e},b}$ the hyperplane

$$H_{\mathbf{e},b} = \{\mathbf{x} \in \mathcal{R}^d : \mathbf{e} \cdot \mathbf{x} + b = 0\}.$$

The closed half-spaces bounded by this hyperplane are denoted by:

$$H^+_{\mathbf{e},b} = \{\mathbf{x} \in \mathcal{R}^d : \mathbf{e} \cdot \mathbf{x} + b \geq 0\}$$

and

$$H^-_{\mathbf{e},b} = \{\mathbf{x} \in \mathcal{R}^d : \mathbf{e} \cdot \mathbf{x} + b \leq 0\}.$$

A function from $\mathcal{R}^d$ into $\mathcal{R}$ is called a *plane wave* if it can be represented in the form $\alpha(\mathbf{v} \cdot \mathbf{x})$, where $\alpha : \mathcal{R} \longrightarrow \mathcal{R}$ is any function of one variable and $\mathbf{v} \in \mathcal{R}^d$ is any fixed nonzero vector. Plane waves are constant on hyperplanes $H_{\mathbf{e},b}$ with $\mathbf{e} = \mathbf{v}/\|\mathbf{v}\|$. A perceptron with activation function ψ thus gives plane waves of the form $\psi_b(\mathbf{v} \cdot \mathbf{x})$, where $\psi_b(t) = \psi(t + b)$.

Our goal is to represent real-valued functions f on $\mathcal{R}^d$ by an integral combination of Heaviside perceptron units. Thus we seek a representation

$$f(\mathbf{x}) = \int_{S^{d-1} \times \mathcal{R}} w(\mathbf{e}, b) \theta(\mathbf{e} \cdot \mathbf{x} + b) \, d\mathbf{e} \, db \tag{2.2}$$

where $d\mathbf{e}$ is the surface area element on the unit sphere S^{d-1}, $w : S^{d-1} \times \mathcal{R} \longrightarrow \mathcal{R}$ is a weight function, and $\mathbf{x} \longmapsto \theta(\mathbf{e} \cdot \mathbf{x} + b)$ is the characteristic function of the closed half-space $H^+_{\mathbf{e},b}$. Equation 2.2 is an integral version of 2.1.

3 Green's functions and Green's second identity

The theory of distributions extends calculus from ordinary differentiable functions to a larger set of *generalized functions* (or *distributions*) where the formal rules of calculus still hold. For example, the operation of convolution $g * h$ of two functions g and h, defined formally by $(g * h)(\mathbf{x}) = \int_{\mathcal{R}^d} g(\mathbf{y}) h(\mathbf{x} - \mathbf{y}) \, d\mathbf{y}$, see [20, p. 123], can be extended to distributions provided their supports are suitable. It is a commutative operation and the Dirac delta function δ serves as an identity with $f * \delta = f$.

Let L be a linear differential operator acting on distributions in $\mathcal{R}^d$. In general the equation

$$L(f) = g$$

with g given and f unknown can be solved for f by means of a *Green's function* G with the property that $L(G) = \delta$. Indeed, $f = g * G$ is a solution since $L(g * G) = g * L(G) = g * \delta = g$.

An example of a linear differential operator is the Laplacian $\triangle$:

$$\triangle g = \sum_{i=1}^{d} \frac{\partial^2 g}{\partial x_i^2}.$$

For a positive integer m, $\triangle^m$ denotes the Laplacian iterated m times, while $\triangle^0$ is the identity operator.

The Green's function for the Laplacian in $\mathcal{R}^d$ is $\frac{1}{2\pi}\log\|\mathbf{x}\|$ when $d = 2$, and $\frac{1}{(2-d)\omega_d}\|\mathbf{x}\|^{2-d}$ when $d \neq 2$, where $\omega_d = \frac{2\sqrt{\pi^d}}{\Gamma(\frac{d}{2})}$ is the surface area of the unit sphere S^{d-1} in $\mathcal{R}^d$ (cf. [5, p. 679]). These Green's functions are *regular* distributions, i.e., they coincide with ordinary functions. Indeed, not only are they locally integrable in $\mathcal{R}^d$ despite a singularity at the origin, they are also C^∞ except at the origin, and they either die out at infinity ($d \geq 3$) or have logarithmic or linear growth there ($d = 2$ or 1). Green's functions for iterated Laplacians, exhibited in Equations (7.3) and (7.4) below, have similar properties but with distinct forms depending on the parity of d, as in the integral formula below.

Our treatment is self-contained and calculus-based. We use distribution theory [20] only for motivation.

Let $\mathbf{x}$ in $\mathcal{R}^d$ be fixed, and let positive numbers δ and R be given with $R \gg \delta$. Let

$$D = \{\mathbf{y} : \mathbf{y} \in \mathcal{R}^d, \|\mathbf{x} - \mathbf{y}\| \geq \delta, \|\mathbf{y}\| \leq R\}$$

so that the boundary of D consists of two spheres: $\partial D = \{\mathbf{y} \in \mathcal{R}^d, \|\mathbf{x} - \mathbf{y}\| = \delta, \text{ or } \|\mathbf{y}\| = R\}$. Then Green's second identity for two C^2 functions u and v (cf. [5, p. 257]) defined in a neighborhood of D takes the form:

$$\int_D (u \triangle v - v \triangle u)\, d\mathbf{y} = \int_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d_S\mathbf{y} \tag{3.1}$$

where $\frac{\partial}{\partial n} = \pm\Sigma_{i=1}^d \frac{y_i}{\|\mathbf{y}\|}\frac{\partial}{\partial y_i}$ denotes the radial normal derivative out of the region of interest and $d_S\mathbf{y}$ denotes the surface area element on the two bounding spheres. We call the righthand side of this equation a *boundary integral*.

4 The Representation Theorem

Let r be a real number. A real-valued function g on $\mathcal{R}^d$ *vanishes to order* r (at infinity) iff $\lim_{\|\mathbf{x}\|\longrightarrow\infty} g(\mathbf{x})\|x\|^r = 0$. In this case we write $g(\mathbf{x}) = o(\|\mathbf{x}\|^{-r})$ according to the Landau convention. Let ord g denote the *order of vanishing* of g at ∞ [2, p. 8], that is,

$$\text{ord } g = \sup\{r : g(\mathbf{x}) = o(\|\mathbf{x}\|^{-r})\}.$$

For example, ord $\frac{(\log(\|\mathbf{x}\|+1))^s}{\|\mathbf{x}\|^{r+1}} = r$ if r and s are real numbers with $s \geq 0$. The statement ord $g > r$ is equivalent to each of the following:

$$\exists\epsilon > 0 \text{ such that } \forall\epsilon_0 > 0\ \exists R > 0 \text{ such that } \|\mathbf{x}\| \geq R \Longrightarrow |g(\mathbf{x})| \leq \frac{\epsilon_0}{\|\mathbf{x}\|^{r+\epsilon}},$$

$$\exists\epsilon > 0\ \exists C > 0 \text{ such that } \forall\mathbf{x}\ |g(\mathbf{x})| \leq \frac{C}{(\|\mathbf{x}\|^2 + 1)^{(r+\epsilon)/2}}, \text{ and}$$

$$\exists\epsilon > 0\ \exists R > 0 \text{ such that } \|\mathbf{x}\| \geq R \Longrightarrow |g(\mathbf{x})| \leq \frac{1}{\|\mathbf{x}\|^{r+\epsilon}}.$$

Lemma 4.1 *Let $g : \mathcal{R}^d \longrightarrow \mathcal{R}$ be C^1, with $\lim_{\|\mathbf{x}\|\longrightarrow\infty} g(\mathbf{x}) = 0$, $\mathbf{x} = (x_1, \ldots, x_d)$, and ord $\frac{\partial g}{\partial x_i} \geq r > 1$ for all $i = 1, \ldots, d$. Then ord $g \geq r - 1$.*

P r o o f. Given $\epsilon_0 > 0$, then for all $\|\mathbf{x}\|$ sufficiently large and for all $i = 1, \dots, d$

$$\left|\frac{\partial g}{\partial x_i}(\mathbf{x})\right| \leq \frac{\epsilon_0}{\|\mathbf{x}\|^r},$$

and

$$\begin{aligned}
|g(\mathbf{x})| &= \left|g(\mathbf{x}) - \lim_{\|\mathbf{x}\| \longrightarrow \infty} g(\mathbf{x})\right| \\
&= \left|\int_{t=1}^{\infty} \sum_{i=1}^{d} \frac{\partial g}{\partial x_i}(t\mathbf{x}) x_i \, dt\right| \\
&\leq \int_{t=1}^{\infty} \frac{\sqrt{d} \cdot \epsilon_0 \|\mathbf{x}\|}{\|t\mathbf{x}\|^r} \, dt \\
&= \frac{\sqrt{d} \cdot \epsilon_0}{\|\mathbf{x}\|^{r-1}} \int_1^{\infty} \frac{dt}{t^r} \\
&= \frac{\sqrt{d} \cdot \epsilon_0}{(r-1)\|\mathbf{x}\|^{r-1}},
\end{aligned}$$

i.e., g vanishes to order $r - 1$. □

Our chief result is that a sufficiently smooth real-valued function on $\mathcal{R}^d$ that dies off at infinity sufficiently rapidly can be written in the form (2.2). The class of functions for which the theorem is proved depends on the parity of d, as does the choice of w.

Let $k_d = d + 1$ if d is odd, $d + 2$ if d is even $\left(\text{i.e., } k_d = 2\lceil \frac{d+1}{2} \rceil\right)$. Call a function f *of controlled decay* if $f : \mathcal{R}^d \longrightarrow \mathcal{R}$ is k_d-times continuously differentiable and for each multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| \leq k_d$

$$\text{ord}\, \partial^\alpha f > |\alpha|$$

where $\partial^\alpha f$ denotes the corresponding partial derivative of f. In other words, for each such α there is an $\epsilon > 0$ such that

$$\lim_{\|\mathbf{x}\| \longrightarrow \infty} \partial^\alpha f(\mathbf{x}) \|\mathbf{x}\|^{|\alpha| + \epsilon} = 0.$$

The number ϵ depends on f and α. In arguments below where multiple partial derivatives are considered, we shall take ϵ to be the smallest of the individual ϵ's and shall also assume without loss of generality that $\epsilon < 1$. The set of functions of controlled decay includes all real-valued functions on $\mathcal{R}^d$ of *rapid descent*, i.e., all $\mathcal{C}^\infty$ functions f satisfying ord $\partial^\alpha f = \infty$ for every multi-index α (cf. [20, p. 100]). In particular, any function in $\mathcal{C}^{k_d}$ with compact support is sufficiently vanishing.

For f of controlled decay, we define a function w_f on $S^{d-1} \times \mathcal{R}$ by

$$w_f(\mathbf{e}, b) = a_d \int_{H^-_{\mathbf{e},b}} \triangle^{\frac{d+1}{2}} f(\mathbf{y}) \, d\mathbf{y} \qquad (d \text{ odd}),$$

$$w_f(\mathbf{e}, b) = a_d \int_{\mathcal{R}^d} \triangle^{\frac{d+2}{2}} f(\mathbf{y}) \alpha(\mathbf{e} \cdot \mathbf{y} + b) \, d\mathbf{y} \qquad (d \text{ even}),$$

where

$$a_d = \begin{cases} \dfrac{(-1)^{(d-1)/2}}{2(2\pi)^{d-1}} & \text{if } d \text{ is odd,} \\[2ex] \dfrac{(-1)^{(d-2)/2}}{(2\pi)^d} & \text{if } d \text{ is even;} \end{cases}$$

and $\alpha(t) = -t \log |t| + t$ for $t \neq 0$ and $\alpha(0) = 0$. The function α is odd.

Our Representation Theorem expresses a function as an integral combination of plane waves based on the Heaviside function θ.

Theorem 4.2 (Representation Theorem) *Let f be of controlled decay. Then for each $\mathbf{x} \in \mathcal{R}^d$*

$$f(\mathbf{x}) = \int_{S^{d-1}\times\mathcal{R}} w_f(\mathbf{e}, b)\vartheta(\mathbf{e}\cdot\mathbf{x} + b)\, d\mathbf{e}\, db, \tag{4.1}$$

where in the even case the integral is defined pointwise as $\lim_{K\longrightarrow\infty}\int_{S^{d-1}\times(-\infty,K]}$.

Well-definedness of w_f and integrability are established in the proof below.

5 The one-dimensional case

The one-dimensional case is instructive both for its simplicity and because it indicates that further generalization is possible. When $d = 1$, any C^1 function that vanishes at $\pm\infty$ can be represented in the form (2.2).

Proposition 5.1 *Let $f : \mathcal{R} \longrightarrow \mathcal{R}$ be continuously differentiable with $\lim_{t\longrightarrow\pm\infty} f(t) = 0$. Then for every $x \in \mathcal{R}$,*

$$f(x) = \frac{1}{2}\int_{-\infty}^{\infty} f'(-b)\vartheta(x + b)\, db + \frac{-1}{2}\int_{-\infty}^{\infty} f'(b)\vartheta(-x + b)\, db.$$

In this case $S^{d-1} = S^0 = \{\pm 1\}$, the weight function is given by $w_f(1, b) = \frac{1}{2}f'(-b) = -w_f(-1,-b)$.

P r o o f. Evaluation of the Heaviside function and a simple change of variables yields:

$$\frac{1}{2}\int_{-\infty}^{x} f'(b)\, db - \frac{1}{2}\int_{x}^{\infty} f'(b)\, db = \frac{1}{2}(f(x) - f(-\infty)) - \frac{1}{2}(f(\infty) - f(x)) = f(x).$$

□

When $d = 1$, the Representation Theorem requires that f be C^2 with sufficient decay at $\pm\infty$, but Proposition 5.1 is more general. The Representation Theorem asserts that when $d = 1$, $w_f(e, b) = a_1 \int_{y:ey+b\le 0} f''(y)\, dy = \frac{1}{2}ef'(-eb)$, as shown.

Proposition 5.1 has alternative forms. The coefficients $\frac{1}{2}$ and $-\frac{1}{2}$ can be replaced by t and $t - 1$ for arbitrary t. If f vanishes only at $-\infty$, take $t = 1$ (the first integral alone). Similarly if f vanishes only at ∞, take $t = 0$. So the weight function in (2.2) is not unique.

Furthermore, f can be assumed to be merely absolutely continuous on each finite interval, guaranteeing a first derivative f' almost everywhere. Assume for example that: i) f is absolutely continuous on finite intervals, ii) $\lim_{x\longrightarrow-\infty} f(x) = 0$, and iii) f has finite total variation on intervals of the form $(-\infty, x]$ for real numbers x. Then f' is in $L^1((-\infty, x])$ for each x and the first integral in Proposition 5.1 represents f at any real number x (cf. Hewitt and Stromberg [12, p. 286]). Lemma 6.1 below shows that the conditions at $\pm\infty$ can be further relaxed.

6 Preliminary lemmas

We need three lemmas for the proof. The first is a variant of Proposition 5.1.

Lemma 6.1 *Let $g : \mathcal{R} \longrightarrow \mathcal{R}$ be a continuous function such that $g(0) = 0$ and g is continuously differentiable on $(-\infty, 0) \cup (0,\infty)$. Then for every $x \in \mathcal{R}$,*

$$g(x) = \int_{0}^{\infty} g'(t)\vartheta(x - t)\, dt - \int_{-\infty}^{0} g'(t)\vartheta(t - x)\, dt.$$

P r o o f. The right side in the Lemma is the same as:

$$\int_{0}^{\max\{x,0\}} g'(t)\, dt - \int_{\min\{0,x\}}^{0} g'(t)\, dt,$$

and this equals $g(x) - g(0) = g(x)$.

□

The function g' is continuous on the half-open interval joining 0 to $x \neq 0$, but may not be integrable on this interval, in which case the integrals in Lemma 6.1 are not Lebesgue or Riemann integrals but rather $\lim_{\epsilon \longrightarrow 0+} \left(\int_{\epsilon}^{\infty} + \int_{-\infty}^{-\epsilon} \right)$. In the two applications we make of Lemma 6.1, g' is integrable on the given intervals and Lebesgue integration applies.

Lemmas 6.2 and 6.3 below show that the functions $\|\mathbf{x}\|$ and $\log \|\mathbf{x}\|$ on $\mathcal{R}^d$ can be represented as integrals of plane waves. These lemmas are stated in [5, pp. 678–679] except for the second part of Lemma 6.3. Lemma 6.2 is needed for d odd, and Lemma 6.3 for d even. We give simple proofs that exploit the homogeneity and rotational invariance of the associated integral formulas. However, extra work is required to establish the final conclusion of Lemma 6.3.

Lemma 6.2 *For every positive integer d and for all $\mathbf{x}$ in $\mathcal{R}^d$,*

$$\|\mathbf{x}\| = s_d \int_{S^{d-1}} |\mathbf{e} \cdot \mathbf{x}| \, d\mathbf{e},$$

where $s_d = \frac{d-1}{2\omega_{d-1}}$ for $d \geq 2$ and $s_1 = \frac{1}{2}$.

Proof. The integral $\int_{\mathbf{e} \in S^{d-1}} |\mathbf{e} \cdot \mathbf{x}| \, d\mathbf{e}$ is positive-homogeneous in $\mathbf{x}$ and rotationally invariant. Thus it is equal to a constant time $\|\mathbf{x}\|$. To evaluate the constant, we take $\mathbf{x}$ to be a unit vector. Then the integral, for $d \geq 2$, is:

$$\begin{aligned} \omega_{d-1} \int_{\theta=0}^{\pi} |\cos\theta| \sin^{d-2}\theta \, d\theta &= \omega_{d-1} \left[\int_0^{\frac{\pi}{2}} \cos\theta \sin^{d-2}\theta \, d\theta - \int_{\frac{\pi}{2}}^{\pi} \cos\theta \sin^{d-2}\theta \, d\theta \right] \\ &= \omega_{d-1} \left[\frac{\sin^{d-1}\theta}{d-1} \bigg|_0^{\frac{\pi}{2}} - \frac{\sin^{d-1}\theta}{d-1} \bigg|_{\frac{\pi}{2}}^{\pi} \right] \\ &= \frac{2\omega_{d-1}}{d-1} \end{aligned}$$

while the integral reduces to 2 when $d = 1$. □

Let $\beta(t) = \frac{t^2}{2} \log |t| - \frac{3t^2}{4}$ for $t \neq 0$ and $\beta(0) = 0$. Note that $\beta'(t) = -\alpha(t)$ for all t, and $\beta''(t) = \log |t|$ for $t \neq 0$.

Lemma 6.3 *For every positive integer d and for all nonzero $\mathbf{x}$ in $\mathcal{R}^d$,*

$$\log \|\mathbf{x}\| = b_d + \frac{1}{\omega_d} \int_{S^{d-1}} \log |\mathbf{e} \cdot \mathbf{x}| \, d\mathbf{e} = b_d + \frac{1}{\omega_d} \triangle \left(\int_{S^{d-1}} \beta(\mathbf{e} \cdot \mathbf{x}) \, d\mathbf{e} \right),$$

where b_d is a constant.

Proof. To show that $\mathbf{e} \longmapsto \log |\mathbf{x} \cdot \mathbf{e}|$ is integrable, it suffices by additivity of the log to assume that $\mathbf{x}$ is a unit vector. If $d = 1$, integrability is trivial. So assume $d \geq 2$. Choosing the direction of $\mathbf{x}$ to be the north pole, letting $\theta = \cos^{-1}(\mathbf{e} \cdot \mathbf{x})$, and using the inequality $\cos\theta \geq 1 - \frac{2}{\pi}\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$, we find that:

$$\begin{aligned} \int_{S^{d-1}} |\log |\mathbf{e} \cdot \mathbf{x}|| \, d\mathbf{e} &= 2\omega_{d-1} \int_0^{\pi/2} |\log\cos\theta| \sin^{d-2}\theta \, d\theta \\ &\leq 2\omega_{d-1} \int_0^{\pi/2} |\log\cos\theta| \, d\theta \\ &\leq 2\omega_{d-1} \int_0^{\pi/2} (-1) \log\left(1 - \frac{2}{\pi}\theta\right) d\theta \\ &= \pi\omega_{d-1}. \end{aligned}$$

Thus the integrals are well-defined despite the singularities at the equator $\theta = \frac{\pi}{2}$.

By rotational invariance ($\mathbf{x}$ no longer assumed to be a unit vector) $\int_{S^{d-1}} \log|\mathbf{e}\cdot\mathbf{x}|\,d\mathbf{e}$ is a function $g(\|\mathbf{x}\|)$, with the property that $g(\|\lambda\mathbf{x}\|) = \omega_d \log|\lambda| + g(\|\mathbf{x}\|)$. Hence $g(\|\mathbf{x}\|) = \omega_d \log\|\mathbf{x}\| + g(1)$, where $g(1) = \omega_{d-1}\int_{\theta=0}^{\pi} \log|\cos\theta|\sin^{d-2}\theta\,d\theta$ for $d \geq 2$ and $g(1) = 0$ when $d = 1$. This establishes the first equality in Lemma 6.3, with $b_d = -g(1)/\omega_d$.

To establish the second equality, observe first that $\mathbf{e} \longmapsto \beta(\mathbf{x}\cdot\mathbf{e})$ is continuous and hence integrable on S^{d-1}. Since $\triangle f(\mathbf{e}\cdot\mathbf{x}) = f''(\mathbf{e}\cdot\mathbf{x})$, we need only show that partial derivatives with respect to $\mathbf{x}$ can be moved from outside to inside the integral. For first derivatives this is clear since β' is continuous and its domain can be restricted to $[-\|\mathbf{x}\| - 1, \|\mathbf{x}\| + 1]$.

The argument for second derivatives is more delicate. Since $\mathbf{e}\cdot(\mathbf{x} + h\mathbf{u}_j) = \mathbf{e}\cdot\mathbf{x} + he_j$, where $\mathbf{u}_j$ is the unit coordinate vector in the j-th direction and $e_j = \mathbf{e}\cdot\mathbf{u}_j$, it suffices to show that

$$\lim_{h\longrightarrow 0}\left(\int_{S^{d-1}}\left(\frac{\beta'(\mathbf{e}\cdot\mathbf{x}+he_j)-\beta'(\mathbf{e}\cdot\mathbf{x})}{h} - e_j\beta''(\mathbf{e}\cdot\mathbf{x})\right)d\mathbf{e}\right) = 0. \tag{6.1}$$

The case $d = 1$ is trivial. Assume $d \geq 2$ and $h > 0$. The integral in (6.1) can be decomposed into the sum of four Lebesgue integrals, two of them over regions in the northern hemisphere:

$$U_h = \{\mathbf{e} : \mathbf{e}\cdot\mathbf{x} > 0, \mathbf{e}\cdot\mathbf{x} + he_j < 0\} \quad \text{and} \quad V_h = \{\mathbf{e} : \mathbf{e}\cdot\mathbf{x} > 0, \mathbf{e}\cdot\mathbf{x} + he_j > 0\},$$

and two more over similar regions in the southern hemisphere.

The set U_h is contained in the set $\{\mathbf{e} : 0 < \mathbf{e}\cdot\mathbf{x} < h\} = \left\{\mathbf{e} : 0 < \cos\theta < \frac{h}{\|\mathbf{x}\|}\right\}$, which has measure approximately equal to $\frac{h\omega_{d-1}}{\|\mathbf{x}\|}$ for h small. It follows that the integral over U_h goes to 0 as h goes to 0 since β' is uniformly continuous on the interval $[-\|\mathbf{x}\| - 1, \|\mathbf{x}\| + 1]$ so that $|\beta'(\mathbf{e}\cdot\mathbf{x} + he_j) - \beta'(\mathbf{e}\cdot\mathbf{x})|$ can be assumed to be arbitrarily small and β'' is integrable.

In the set V_h, on the other hand, let τ be the absolute value of the integrand in (6.1). Setting $t = \frac{he_j}{\mathbf{e}\cdot\mathbf{x}}$ and using the definitions of β' and β'', we find that

$$\tau = |e_j|\,\text{sign(t)}\left(\frac{\log(1+t)}{t} + \log(1+t) - 1\right).$$

If $t = 0$, $\tau = 0$. Since $|e_j| \leq \|\mathbf{e}\| = 1$, τ is dominated by $\log(1+t)$ for $t \geq 0$ and by $\frac{\log(1-t)}{\log 2}$ for $-1 < t \leq 0$. Thus τ is dominated by $\frac{\log(1+|t|)}{\log 2}$ for all $t > -1$. Hence, with $H = \frac{h}{\|\mathbf{x}\|}$, the integral over V_h is dominated by:

$$\begin{aligned}\frac{1}{\log 2}\int_{\{\mathbf{e}:\,\mathbf{e}\cdot\mathbf{x}>0\}} \log\left(1 + h\left|\frac{e_j}{\mathbf{e}\cdot\mathbf{x}}\right|\right) d\mathbf{e} &\leq \frac{\omega_{d-1}}{\log 2}\int_0^{\pi/2} \log\left(1 + \frac{H}{\cos\theta}\right)\sin^{d-2}\theta\,d\theta \\ &\leq \frac{\omega_{d-1}}{\log 2}\int_0^{\pi/2} \log\left(1 + \frac{H}{1-\frac{2}{\pi}\theta}\right) d\theta \\ &= \frac{\omega_{d-1}}{\log 2}\,\frac{\pi}{2}\left(-H\log H + (1+H)\log(1+H)\right),\end{aligned}$$

which converges to 0 as h tends to 0^+.

These arguments show that when h converges to 0^+ the integral in (6.1) restricted to the northern hemisphere ($\mathbf{e}$ such that $\mathbf{e}\cdot\mathbf{x} > 0$) tends to 0. In the southern hemisphere the substitution $\mathbf{e}' = -\mathbf{e}$ and the oddness of β' and evenness of β'' convert the corresponding integral into the negative of the northern hemisphere integral. Likewise, when h tends to 0 though negative values, the substitutions $\mathbf{x}' = -\mathbf{x}$ and $h' = -h$ convert a left limit at $\mathbf{x}$ into a right limit at $\mathbf{x}'$. Hence (6.1) holds. □

7 Proof of the Representation Theorem

P r o o f. To prove the Representation Theorem, we express a function as the convolution of its iterated Laplacian with the corresponding Green's function. The Green's function is represented as an integral combination of plane waves. The plane waves in turn are represented as integral combinations of Heavisides. This gives the desired representation of the original function.

Throughout the argument $\mathbf{x}$ is an arbitrary fixed member of $\mathcal{R}^d$, while $\mathbf{y}$ in $\mathcal{R}^d$ is variable and $\triangle = \triangle_{\mathbf{y}}$.

First we show that w_f is finite and continuous. Indeed, the integrands in the definition of w_f in Section 4 are sums involving derivatives of order k_d since they contain iterated Laplacians.

In the odd case, when $|\alpha| = k_d = d+1$, the summands satisfy

$$|\partial^\alpha f(\mathbf{y})| \le \frac{C}{(\|\mathbf{y}\|^2+1)^{(d+1+\epsilon)/2}}$$

for all $\mathbf{y}$ in $\mathcal{R}^d$ and some $\epsilon > 0$ where C is a constant depending on f and α. Thus in spherical coordinates, where $r = \| \mathbf{y} \|$ and $d\Omega$ is the area element on S^{d-1}, integrating over all of $\mathcal{R}^d$ rather than just $H^-_{\mathbf{e},b}$, we find:

$$\begin{aligned}\int_{\mathcal{R}^d} |\partial^\alpha f(\mathbf{y})|\, d\mathbf{y} &\le \int_{[0,\infty)\times S^{d-1}} \frac{C}{(r^2+1)^{(d+1+\epsilon)/2}} r^{d-1}\, d\Omega\, dr \\ &\le C\omega_d \int_0^\infty \frac{dr}{(r^2+1)^{(2+\epsilon)/2}} \\ &\le C\omega_d \frac{\pi}{2}.\end{aligned}$$

Hence, for d odd, w_f is finite and Lebesgue's Dominated Convergence Theorem [12, p. 172] applied to the effective integrand $\triangle^{\frac{d+1}{2}} f(\mathbf{y})\theta(-\mathbf{e}\cdot\mathbf{y} - b)$ shows that w_f is continuous.

In the even case, when $|\alpha| = k_d = d+2$, the summands satisfy

$$|\partial^\alpha f(\mathbf{y})||\alpha(\mathbf{e}\cdot\mathbf{y}+b)| \le \frac{C|\alpha(\mathbf{e}\cdot\mathbf{y}+b)|}{(\|\mathbf{y}\|^2+1)^{(d+2+\epsilon)/2}}$$

for $\|\mathbf{y}\|$ in $\mathcal{R}^d$. If we adopt a coordinate system in which $\mathbf{y} = (y_1, \mathbf{y}^\perp)$ with $y_1 = \mathbf{e}\cdot\mathbf{y}$ and $\mathbf{y}^\perp \in \mathcal{R}^{d-1}$, and $\rho = \|\mathbf{y}^\perp\|$, we obtain:

$$\int_{\mathcal{R}^d} |\partial^\alpha f(\mathbf{y})||\alpha(\mathbf{e}\cdot\mathbf{y}+b)|\, d\mathbf{y} \le \int_{\mathcal{R}\times[0,\infty)\times S^{d-2}} \frac{C|\alpha(y_1+b)|}{(y_1^2+\rho^2+1)^{(d+2+\epsilon)/2}} \rho^{d-2}\, d\Omega\, d\rho\, dy_1.$$

We make the substitution $\rho = \sqrt{y_1^2+1}\tan\theta$, so that $d\rho = \sqrt{y_1^2+1}\sec^2\theta\, d\theta$ where sec denotes *Sekans*, and the right side becomes:

$$\begin{aligned}&C\omega_{d-1}\int_{\mathcal{R}}\int_{\theta=0}^{\pi/2} \frac{|\alpha(y_1+b)|(y_1^2+1)^{(d-1)/2}}{(y_1^2+1)^{(d+2+\epsilon)/2}} \frac{\tan^{d-2}\theta\sec^2\theta\, d\theta\, dy_1}{\sec^{d+2+\epsilon}\theta} \\ &\quad\le C\omega_{d-1}\frac{\pi}{2}\int_{\mathcal{R}} \frac{|\alpha(y_1+b)|\, dy_1}{(y_1^2+1)^{(3+\epsilon)/2}}.\end{aligned}$$

The last integral is an even function of b: replace b by $-b$ and y_1 by $-y_1$. So we may assume $b \ge 0$. Since $|\alpha(t)| \le 1$ for $|t| \le e+1$, this integral is dominated by the following sum:

$$\int_{|y_1+b|\le e+1} \frac{dy_1}{(y_1^2+1)^{(3+\epsilon)/2}} + \int_{|y_1+b|\ge e+1, |y_1|\le b} \frac{|\alpha(y_1+b)|\, dy_1}{(y_1^2+1)^{(3+\epsilon)/2}} + \int_{|y_1+b|\ge e+1, |y_1|\ge b} \frac{|\alpha(y_1+b)|\, dy_1}{(y_1^2+1)^{(3+\epsilon)/2}}.$$

Since $e \le |t_1| \le |t_2|$ implies $|\alpha(|t_1|)| \le |\alpha(|t_2|)|$ and in the second and third integrals immediately above $e+1 \le |y_1+b| \le |y_1| + b \le \max\{2|y_1|, 2b\}$, the sum is dominated by:

$$\int_{\mathcal{R}} \frac{(1+|\alpha(2b)| + |\alpha(2|y_1|)|)\, dy_1}{(y_1^2+1)^{(3+\epsilon)/2}}.$$

Since the integrands in these successive inequalities are continuous functions of b, or of b and $\mathbf{e}$, and the last integral is finite and continuously dependent on b, Lebesgue's Dominated Convergence Theorem applies as in the odd case, and so w_f is finite and continuous.

7.1 The case of odd d

7.1.1 Finding the Green's function

For $i = 0, 1, \ldots, \frac{d+1}{2}$ let

$$u_i = \triangle^i f, \quad v_i = \triangle^{\frac{d+1}{2}-i} \parallel \mathbf{x} - \mathbf{y} \parallel,$$

where differentiation in the Laplacians is with respect to the variable $\mathbf{y}$. Then for $0 < i < \frac{d+1}{2}$, u_i and v_{i+1} are twice continuously differentiable except when $\mathbf{y} = \mathbf{x}$ since f is in C^{d+1}, and we assert that for such i:

$$\int_{\mathcal{R}^d} u_i v_i \, d\mathbf{y} = \int_{\mathcal{R}^d} u_i \triangle v_{i+1} \, d\mathbf{y} = \int_{\mathcal{R}^d} \triangle u_i v_{i+1} \, d\mathbf{y} = \int_{\mathcal{R}^d} u_{i+1} v_{i+1} \, d\mathbf{y}. \tag{7.1}$$

The middle equation is based on Green's identity for u_i, v_{i+1} on the region D in Section 3, Equation (3.1), and requires that the corresponding boundary integrals vanish as R tends to ∞ and δ tends to 0. These conditions will be established below.

First we must show that $u_i v_i$ is integrable for $0 \leq i \leq \frac{d+1}{2}$. Since $u_i v_i$ is continuous, we only need investigate behavior as $\mathbf{y}$ tends to ∞ or $\mathbf{y}$ tends to $\mathbf{x}$. We make extensive use of the following identities:

$$\begin{aligned} &\triangle \left(\parallel \mathbf{x} - \mathbf{y} \parallel^a\right) = a(a+d-2) \parallel \mathbf{x} - \mathbf{y} \parallel^{a-2} \\ &\triangle^m \left(\parallel \mathbf{x} - \mathbf{y} \parallel^a\right) = C(a,m,d) \parallel \mathbf{x} - \mathbf{y} \parallel^{a-2m} \\ &C(a,m,d) = \Pi_{j=0}^{m-1}(a-2j)\Pi_{j=1}^{m}(a+d-2j), \end{aligned} \tag{7.2}$$

valid for $\mathbf{y} \neq \mathbf{x}$, any real number a, and any integer $m > 0$. The decay condition on f assumed in the theorem provides an $\epsilon > 0$ such that

$$|u_i(\mathbf{y})| \leq \frac{1}{\parallel \mathbf{y} \parallel^{2i+\epsilon}}$$

for $\parallel \mathbf{y} \parallel$ sufficiently large. Using the identities (7.2), we find that

$$|v_i(\mathbf{y})| = \left| C\left(1, \frac{d+1}{2} - i, d\right) \right| \parallel \mathbf{x} - \mathbf{y} \parallel^{1-2(\frac{d+1}{2}-i)}$$

for $\mathbf{y} \neq \mathbf{x}$. Since $\lim_{\|\mathbf{y}\| \longrightarrow \infty} \frac{\|\mathbf{y}\|}{\|\mathbf{x}-\mathbf{y}\|} = 1$, with A any real number larger than $C\left(1, \frac{d+1}{2} - i, d\right)$ and R sufficiently large, we obtain in spherical coordinates centered on the origin:

$$\begin{aligned} \int_{\{\mathbf{y}\,:\|\mathbf{y}\| \geq R\}} |u_i v_i| \, d\mathbf{y} &\leq \int_{[R,\infty) \times S^{d-1}} \frac{A}{r^{2i+\epsilon}} r^{1-2(\frac{d+1}{2}-i)} r^{d-1} \, dr \, d\Omega \\ &= \int_{[R,\infty) \times S^{d-1}} \frac{A}{r^{1+\epsilon}} \, dr \, d\Omega \\ &= \frac{A\omega_d}{\epsilon R^\epsilon} \\ &< \infty. \end{aligned}$$

Likewise, for $\delta > 0$ and $i > 0$, in spherical cordinates centered on $\mathbf{x}$, we have:

$$\begin{aligned} \int_{\{\mathbf{y}\,:0 < \|\mathbf{x}-\mathbf{y}\| \leq \delta\}} |u_i v_i| \, d\mathbf{y} &\leq \int_{(0,\delta] \times S^{d-1}} B r^{1-2(\frac{d+1}{2}-i)} r^{d-1} \, dr \, d\Omega \\ &= \int_{(0,\delta] \times S^{d-1}} B r^{2i-1} \, dr \, d\Omega \\ &= \frac{B\delta^{2i}\omega_d}{2i} < \infty \end{aligned}$$

where $B = \max\{|u_i(\mathbf{y})| : \| \mathbf{x}-\mathbf{y} \| \le \delta\} C\left(1, \frac{d+1}{2}-i, d\right)$. When $i = 0$, there is nothing to prove since $C\left(1, \frac{d+1}{2}, d\right) = 0$ and so $v_0 \equiv 0$. Thus $u_i v_i$ is integrable as claimed.

Now we show that the boundary integrals vanish. For R sufficiently large there is a constant C such that

$$\begin{aligned}\int_{\{\mathbf{y}\,:\|\mathbf{y}\|=R\}} \left| u_i \frac{\partial v_{i+1}}{\partial n} - v_{i+1}\frac{\partial u_i}{\partial n}\right| d_S\mathbf{y} &\le \int_{S^{d-1}} \frac{C}{R^{2i+\epsilon}} R^{1-2(\frac{d+1}{2}-(i+1))-1} R^{d-1}\, d\Omega \\ &= \int_{S^{d-1}} \frac{C\, d\Omega}{R^{\epsilon}} \\ &= \frac{C\omega_d}{R^{\epsilon}},\end{aligned}$$

where ϵ is from the decay condition on f. Indeed, $\frac{\partial}{\partial n} = \pm\Sigma_{i=1}^{d} \frac{y_i}{r}\frac{\partial}{\partial y_i}$ and $\left|\frac{\partial}{\partial n}(\| \mathbf{x}-\mathbf{y} \|^a)\right| \le a \| \mathbf{x}-\mathbf{y} \|^{a-1}$ so that

$$\left| u_i(\mathbf{y})\frac{\partial v_i}{\partial n}(\mathbf{y})\right| \le \frac{1}{R^{2i+\epsilon}} C_1 \| \mathbf{x}-\mathbf{y} \|^{1-2(\frac{d+1}{2}-i)-1}, \quad \text{and}$$

$$\left| v_{i+1}(\mathbf{y})\frac{\partial u_i}{\partial n}(\mathbf{y})\right| \le \frac{C_2 \| \mathbf{x}-\mathbf{y} \|^{1-2(\frac{d+1}{2}-i)}}{R^{2i+\epsilon+1}}$$

for suitable constants C_1 and C_2 depending on i and d. Since $\frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{y}\|} \sim 1$ for $\| \mathbf{y} \|$ sufficiently large, we can take $C > C_1 + C_2$ to obtain the above estimate on the boundary integral for large R. For δ near 0, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$ where r is the radial coordinate with center $\mathbf{x}$, and

$$\begin{aligned}&\int_{\{\mathbf{y}\,:\|\mathbf{x}-\mathbf{y}\|=\delta\}} \left| u_i \frac{\partial v_{i+1}}{\partial n} - v_{i+1}\frac{\partial u_i}{\partial n}\right| d_S\mathbf{y} \\ &\quad\le \int_{S^{d-1}} \left\{ D_1 \delta^{1-2(\frac{d+1}{2}-(i+1))-1} + D_2 \delta^{1-2(\frac{d+1}{2}-(i+1))}\right\} \delta^{d-1}\, d\Omega \\ &\quad= \left\{ D_1\delta^{2i} + D_2\delta^{2i+1}\right\}\omega_d\end{aligned}$$

where

$$\begin{aligned}D_1 &= \max\{|u_i(\mathbf{y})| : \| \mathbf{x}-\mathbf{y} \| \le \delta\}\cdot C\left(1, \frac{d+1}{2}-(i+1), d\right)|2(i+1)-d| \quad \text{and} \\ D_2 &= \max\left\{\left|\frac{\partial u_i}{\partial n}(\mathbf{y})\right| : \| \mathbf{x}-\mathbf{y} \| \le \delta\right\}\cdot C\left(1, \frac{d+1}{2}-(i+1), d\right).\end{aligned}$$

As R tends to ∞ and δ to 0^+, these estimates show that all boundary integrals vanish for $0 < i < \frac{d+1}{2}$. This establishes (7.1).

When $i = 0$, the boundary integrals vanish with one exception:

$$\begin{aligned}\int_{\{\mathbf{y}\,:\|\mathbf{y}-\mathbf{x}\|=\delta\}} u_0 \frac{\partial v_1}{\partial n}\, d_S\mathbf{y} &= -\int_{S^{d-1}} f(\mathbf{y}) \frac{\partial \triangle^{\frac{d-1}{2}}(\| \mathbf{x}-\mathbf{y} \|)}{\partial r}\bigg|_{r=\delta} \delta^{d-1}\, d\Omega \\ &= -\int_{S^{d-1}} f(\mathbf{y})\frac{\partial}{\partial r}\left(C\left(1, \frac{d+1}{2}-1, d\right) \| \mathbf{x}-\mathbf{y} \|^{1-2(\frac{d+1}{2}-1)}\right)\bigg|_{r=\delta} \delta^{d-1}\, d\Omega \\ &= -\int_{S^{d-1}} f(\mathbf{y}) C\left(1, \frac{d+1}{2}-1, d\right)(2-d)\delta^{1-d}\delta^{d-1}\, d\Omega \\ &= \int_{S^{d-1}} f(\mathbf{y})(-1)^{\frac{d+1}{2}}(d-1)!\,\delta^{1-d}\delta^{d-1}\, d\Omega,\end{aligned}$$

where $\mathbf{y} = \mathbf{x} + \delta\mathbf{e}$, $\mathbf{e} \in S^{d-1}$, and this integral tends to $f(\mathbf{x})(-1)^{\frac{d+1}{2}}(d-1)!\,\omega_d$ as δ tends to 0.

By (7.1)

$$\int_{\mathcal{R}^d} u_1 v_1 \, d\mathbf{y} = \cdots = \int_{\mathcal{R}^d} u_{\frac{d+1}{2}} v_{\frac{d+1}{2}} \, d\mathbf{y},$$

and by Green's identity again and the fact that $v_0 \equiv 0$

$$\begin{aligned} 0 &= \int_{\mathcal{R}^d} u_0 v_0 \, d\mathbf{y} \\ &= \int_{\mathcal{R}^d} u_1 v_1 \, d\mathbf{y} + \lim_{R \longrightarrow \infty, \delta \longrightarrow 0^+} \int_{\{\mathbf{y} : \|\mathbf{x}-\mathbf{y}\|=\delta \text{or} \|\mathbf{y}\|=R\}} u_0 \left(\frac{\partial v_1}{\partial n} - v_1 \frac{\partial u_0}{\partial n} \right) d_S \mathbf{y} \\ &= \int_{\mathcal{R}^d} u_{\frac{d+1}{2}} v_{\frac{d+1}{2}} \, d\mathbf{y} - f(\mathbf{y})(-1)^{\frac{d-1}{2}} (d-1)! \, \omega_d. \end{aligned}$$

Hence

$$f(\mathbf{x}) = c_d \int_{\mathcal{R}^d} \triangle^{\frac{d+1}{2}} f(\mathbf{y}) \|\mathbf{x} - \mathbf{y}\| \, d\mathbf{y} \tag{7.3}$$

with $c_d = \frac{(-1)^{\frac{d-1}{2}}}{(d-1)! \, \omega_d}$. So $c_d \parallel \mathbf{y} \parallel$ is the Green's function for $\triangle^{\frac{d+1}{2}}$ in $\mathcal{R}^d$ (d odd).

Determining w_f. By Lemmas 6.2 and 6.1, applied to the function $|t|$, we have:

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &= s_d \int_{S^{d-1}} |\mathbf{e} \cdot (\mathbf{x} - \mathbf{y})| \, d\mathbf{e} \\ &= s_d \int_{S^{d-1}} \left(\int_0^\infty \vartheta(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}) - t) \, dt + \int_{-\infty}^0 \vartheta(t - \mathbf{e} \cdot (\mathbf{x} - \mathbf{y})) \, dt \right) d\mathbf{e}. \end{aligned}$$

Now set $b = -\mathbf{e} \cdot \mathbf{y} - t$ in the first integral and $b = \mathbf{e} \cdot \mathbf{y} + t$ in the second, and use the identity $\int_{S^{d-1}} g(-\mathbf{e}) \, d\mathbf{e} = \int_{S^{d-1}} g(\mathbf{e}) \, d\mathbf{e}$ (true for any integrable function g). Then

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &= s_d \int_{S^{d-1}} \left(\int_{-\infty}^{-\mathbf{e} \cdot \mathbf{y}} \vartheta(\mathbf{e} \cdot \mathbf{x} + b) \, db + \int_{-\infty}^{\mathbf{e} \cdot \mathbf{y}} \vartheta(-\mathbf{e} \cdot \mathbf{x} + b) \, db \right) d\mathbf{e} \\ &= 2 s_d \int_{S^{d-1}} \int_{-\infty}^{-\mathbf{e} \cdot \mathbf{y}} \vartheta(\mathbf{e} \cdot \mathbf{x} + b) \, db \, d\mathbf{e} \\ &= 2 s_d \int_{S^{d-1}} \int_{\mathcal{R}} \vartheta(-\mathbf{e} \cdot \mathbf{y} - b) \vartheta(\mathbf{e} \cdot \mathbf{x} + b) \, db \, d\mathbf{e}. \end{aligned}$$

Substituting this expression for $\parallel \mathbf{x} - \mathbf{y} \parallel$ into (7.3), and supposing that we may change the order of integration, we obtain the representation of Theorem 4.2 for d odd:

$$\begin{aligned} f(\mathbf{x}) &= 2 c_d s_d \int_{S^{d-1}} \int_{\mathcal{R}} \left(\int_{\mathcal{R}^d} \triangle^{\frac{d+1}{2}} f(\mathbf{y}) \vartheta(-\mathbf{e} \cdot \mathbf{y} - b) \, d\mathbf{y} \right) \vartheta(\mathbf{x} \cdot \mathbf{e} + b) \, d\mathbf{e} \, db \\ &= \int_{S^{d-1} \times \mathcal{R}} w_f(\mathbf{e}, b) \vartheta(\mathbf{x} \cdot \mathbf{e} + b) \, db \, d\mathbf{e}. \end{aligned}$$

Change in the order of integration is justified by Fubini's Theorem (cf. [12, p. 386]). For each $\mathbf{x}$ in $\mathcal{R}^d$, $(\mathbf{y}, \mathbf{e}, b) \longmapsto \triangle^{\frac{d+1}{2}} f(\mathbf{y}) \vartheta(-\mathbf{e} \cdot \mathbf{y} - b) \vartheta(\mathbf{e} \cdot \mathbf{x} + b)$ is integrable with absolute integral

$$\begin{aligned} &\int_{\mathcal{R}^d} \int_{S^{d-1}} \int_{\mathcal{R}} \left| \triangle^{\frac{d+1}{2}} f(\mathbf{y}) \right| |\vartheta(-\mathbf{e} \cdot \mathbf{y} - b) \vartheta(\mathbf{e} \cdot \mathbf{x} + b)| \, db \, d\mathbf{e} \, d\mathbf{y} \\ &\quad \le \int_{\mathcal{R}^d} \int_{S^{d-1}} \left| \triangle^{\frac{d+1}{2}} f(\mathbf{y}) \right| |\mathbf{e} \cdot (\mathbf{x} - \mathbf{y})| \, d\mathbf{e} \, d\mathbf{y} \le \omega_d \int_{\mathcal{R}^d} | \triangle^{\frac{d+1}{2}} f(\mathbf{y}) | \parallel \mathbf{x} - \mathbf{y} \parallel \, d\mathbf{y}, \end{aligned}$$

which is bounded since $\lim_{\|\mathbf{y}\|\longrightarrow\infty}\frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{y}\|}=1$ and for large R:

$$\int_{\{\mathbf{y}\,:\|\mathbf{y}\|\geq R\}}\left|\triangle^{\frac{d+1}{2}}f(\mathbf{y})\right|\,\|\,\mathbf{y}\,\|\;d\mathbf{y}\leq\int_{\|\mathbf{y}\|\geq R}\frac{1}{r^{d+1+\epsilon}}r\cdot r^{d-1}\,dr\,d\Omega=\int_{\|\mathbf{y}\|\geq R}\frac{dr\,d\Omega}{r^{1+\epsilon}}=\frac{\omega_d}{\epsilon R^\epsilon}.$$

In particular, by Fubini the intermediate integrand $w_f(\mathbf{e},b)\vartheta(\mathbf{e}\cdot\mathbf{x}+b)$ is integrable on $S^{d-1}\times\mathcal{R}$.

7.2 The case of even d

The argument in this case is similar to the odd case except that we construct both w_f and an alternative weight function $\hat{w}_f$.

7.2.1 Finding the Green's function

Proceeding as in the odd case, we let

$$u_i=\triangle^i f,\quad v_i=\triangle^{\frac{d}{2}-i}(\log\|\,\mathbf{x}-\mathbf{y}\,\|)$$

for $0\leq i\leq\frac{d}{2}$. We shall establish that equations (7.1) hold for these choices of u_i and v_i and $0<i<\frac{d}{2}$.

We shall need the following identities:

$$\begin{aligned}&\triangle\,(\log\|\,\mathbf{x}-\mathbf{y}\,\|)=(d-2)\,\|\,\mathbf{x}-\mathbf{y}\,\|^{-2},\\&\triangle^m\,(\log\|\,\mathbf{x}-\mathbf{y}\,\|)=C(-2,m-1,d)(d-2)\,\|\,\mathbf{x}-\mathbf{y}\,\|^{-2m}\quad\text{for}\quad m\geq2,\end{aligned}\tag{7.2'}$$

where $C(\cdot,\cdot,\cdot)$ is as in (7.2). These identities imply that $v_0(\mathbf{y})=\triangle^{\frac{d}{2}}(\|\,\mathbf{x}-\mathbf{y}\,\|)\equiv0$.

The function u_iv_i is integrable for $0\leq i\leq\frac{d}{2}$. Indeed,

$$\begin{aligned}\int_{\{\mathbf{y}\,:\|\mathbf{y}\|\geq R\}}|u_iv_i|\,d\mathbf{y}&\leq\int_{[R,\infty)\times S^{d-1}}\frac{A}{r^{2i+\epsilon}}r^{-2(\frac{d}{2}-i)}r^{d-1}\,dr\,d\Omega\\&=\int_{[R,\infty)\times S^{d-1}}\frac{A\,dr\,d\Omega}{r^{1+\epsilon}}\\&=\frac{A\omega_d}{\epsilon R^\epsilon}\end{aligned}$$

for $r=\|\,\mathbf{y}\,\|\geq R$ with R large, $i<\frac{d}{2}$, and a suitable constant A depending on i and d. For $i=\frac{d}{2}$ and R large

$$\begin{aligned}\int_{\{\mathbf{y}\,:\|\mathbf{y}\|\geq R\}}|u_{\frac{d}{2}}v_{\frac{d}{2}}|\,d\mathbf{y}&\leq\int_{[R,\infty)\times S^{d-1}}\frac{1}{r^{d+\epsilon}}(\log r)r^{d-1}\,dr\,d\Omega\\&=\int_{[R,\infty)\times S^{d-1}}\frac{\log r}{r^{1+\epsilon}}\,dr\,d\Omega\\&=\frac{(1+\epsilon\log R)\omega_d}{\epsilon^2R^\epsilon}.\end{aligned}$$

For $\delta>0$ and $0<i<\frac{d}{2}$,

$$\begin{aligned}\int_{\{\mathbf{y}\,:0<\|\mathbf{x}-\mathbf{y}\|\leq\delta\}}|u_iv_i|\,d\mathbf{y}&\leq\int_{(0,\delta]\times S^{d-1}}Br^{-2(\frac{d}{2}-i)}r^{d-1}\,dr\,d\Omega\\&=\int_{(0,\delta]\times S^{d-1}}Br^{2i-1}\,dr\,d\Omega\\&=\frac{B\delta^{2i}\omega_d}{2i},\end{aligned}$$

where $B = \max\{| \triangle^i f(\mathbf{y})| : \| \mathbf{x} - \mathbf{y} \| \le \delta\} \cdot A$ with A as above. When $i = \frac{d}{2}$ and $\delta < 1$,

$$\int_{\{\mathbf{y}\,:0<\|\mathbf{x}-\mathbf{y}\|\le\delta\}} \left|u_{\frac{d}{2}} v_{\frac{d}{2}}\right| d\mathbf{y} \le \int_{(0,\delta]\times S^{d-1}} K|\log r| r^{d-1}\, dr\, d\Omega = \frac{K\delta^d |d\log\delta - 1|\omega_d}{d^2},$$

where $K = \max\left\{\left|\triangle^{\frac{d}{2}} f(\mathbf{y})\right| \,:\, \| \mathbf{x} - \mathbf{y} \| \le \delta\right\}$. Thus, $u_i v_i$ is integrable.

We now establish the vanishing of the boundary integrals in Green's second identity to show that (7.1) holds. For $0 \le i < \frac{d}{2}$ and large R

$$\begin{aligned}\int_{\{\mathbf{y}\,:\|\mathbf{y}\|=R\}} \left|u_i \frac{\partial v_{i+1}}{\partial n} - v_{i+1}\frac{\partial u_i}{\partial n}\right| d_S\mathbf{y} &\le \int_{S^{d-1}} \frac{C}{R^{2i+\epsilon}} R^{-2(\frac{d}{2}-(i+1))-1} R^{d-1}\, d\Omega \\ &= \int_{S^{d-1}} \frac{C\, d\Omega}{R^{\epsilon}} \\ &= \frac{C\omega_d}{R^{\epsilon}},\end{aligned}$$

where C is a constant depending on i and d (obtained by combining two constants C_1 and C_2 for the two terms as in the odd case). For δ near 0

$$\begin{aligned}&\int_{\{\mathbf{y}\,:\|\mathbf{x}-\mathbf{y}\|=\delta\}} \left|u_i \frac{\partial v_{i+1}}{\partial n} - v_{i+1}\frac{\partial u_i}{\partial n}\right| d_S\mathbf{y} \\ &\quad\le \int_{S^{d-1}} \left\{D_1 \delta^{-2(\frac{d}{2}-(i+1))-1} + D_2\delta^{-2(\frac{d}{2}-(i+1))}\right\} \delta^{d-1}\, d\Omega \\ &\quad= \left\{D_1\delta^{2i} + D_2\delta^{2i+1}\right\}\omega_d\end{aligned}$$

where D_1 and D_2 are suitable constants as in the odd case. As R tends to ∞ and δ goes to 0^+, the boundary integrals tend to 0 and so (7.1) is valid.

For $i = 0$, as in the case of odd d, there is one non-negligible term in the boundary integral. By (7.2')

$$\begin{aligned}\int_{\{\mathbf{y}\,:\|\mathbf{x}-\mathbf{y}\|=\delta\}} u_0 \frac{\partial v_1}{\partial n}\, d_S\mathbf{y} &= -\int_{S^{d-1}} f(\mathbf{y}) \frac{\partial}{\partial r}\bigg|_{r=\delta} \triangle^{\frac{d-2}{2}} (\log \| \mathbf{x} - \mathbf{y} \|)\delta^{d-1}\, d\Omega \\ &= -\int_{S^{d-1}} f(\mathbf{y}) C\left(-2, \frac{d-2}{2} - 1, d\right)(d-2)\frac{\partial}{\partial r}\bigg|_{r=\delta} (r^{2-d})\delta^{d-1}\, d\Omega \\ &= \int_{S^{d-1}} f(\mathbf{y})(-1)^{\frac{d}{2}} \left\{\left(\frac{d-2}{2}\right)!\right\}^2 2^{d-2}\delta^{1-d}\delta^{d-1}\, d\Omega,\end{aligned}$$

and this integral tends to $f(\mathbf{x})(-1)^{\frac{d}{2}}\{(\frac{d-2}{2})!\}^2 2^{d-2}\omega_d$ as δ tends to 0. As in the odd case, cf. (7.3),

$$\lim_{\delta\longrightarrow 0+} -\int_{\{\mathbf{y}\,:\,\|\mathbf{x}-\mathbf{y}\|=\delta\}} u_0\frac{\partial v_1}{\partial n}\, d_S\mathbf{y} = \int_{\mathcal{R}^d} u_1 v_1\, d\mathbf{y} = \int_{\mathcal{R}^d} u_{\frac{d}{2}} v_{\frac{d}{2}}\, d\mathbf{y},$$

and so

$$f(\mathbf{x}) = c_d \int_{\mathcal{R}^d} \triangle^{\frac{d}{2}} f(\mathbf{y}) \log\|\mathbf{x} - \mathbf{y}\|\, d\mathbf{y} \tag{7.4}$$

where $c_d = \frac{(-1)^{\frac{d-2}{2}}}{2^{d-2}\omega_d\{(\frac{d-2}{2})!\}^2}$. Thus $c_d \log\|\mathbf{y}\|$ is the Green's function for $\triangle^{\frac{d}{2}}$ in $\mathcal{R}^d$ (d even).

Determining $\hat{w}_f$. By Lemma 6.3

$$\log\|\mathbf{x} - \mathbf{y}\| = b_d + \frac{1}{\omega_d}\triangle\left(\int_{S^{d-1}} \beta(\mathbf{e}\cdot(\mathbf{x}-\mathbf{y}))\, d\mathbf{e}\right). \tag{7.5}$$

The Divergence Theorem [3, p. 423] implies that

$$\int_{\mathcal{R}^d} \triangle^{\frac{d}{2}} f(\mathbf{y})\, d\mathbf{y} = \lim_{R \longrightarrow \infty} \int_{\{\mathbf{y}\,:\, \|\mathbf{y}\|\, = R\}} \frac{\partial}{\partial n}\big(\triangle^{\frac{d-2}{2}} f\big)(\mathbf{y}) d_S \mathbf{y} = 0$$

since

$$\int_{\{\mathbf{y}\,:\, \|\mathbf{y}\| = R\}} \left| \frac{\partial}{\partial n}\big(\triangle^{\frac{d-2}{2}} f\big)(\mathbf{y}) \right| d_S \mathbf{y} \le \int_{\{\mathbf{y}\,:\, \|\mathbf{y}\| = R\}} \frac{1}{R^{2(\frac{d-2}{2})+1+\epsilon}} R^{d-1}\, d\Omega = \frac{\omega_d}{R^\epsilon}$$

for large R. Using this fact and substituting (7.5) into (7.4), we can eliminate the term involving b_d to obtain:

$$f(\mathbf{x}) = \frac{c_d}{\omega_d} \int_{\mathcal{R}^d} \triangle^{\frac{d}{2}} f(\mathbf{y}) \triangle \left(\int_{S^{d-1}} \beta(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))\, d\mathbf{e} \right) d\mathbf{y}.$$

Then application of Green's identity converts this into:

$$f(\mathbf{x}) = \frac{c_d}{\omega_d} \int_{\mathcal{R}^d} \triangle^{\frac{d+2}{2}} f(\mathbf{y}) \left(\int_{S^{d-1}} \beta(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))\, d\mathbf{e} \right) d\mathbf{y}. \tag{7.6}$$

Indeed, to apply (3.1), let $u(\mathbf{y}) = \triangle^{\frac{d}{2}} f(\mathbf{y})$ and $v(\mathbf{y}) = \int_{S^{d-1}} \beta(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))\, d\mathbf{e}$. Then the boundary integral in (3.1) decomposes into two parts, one an integral over $\{\mathbf{y} : \ \| \mathbf{x} - \mathbf{y} \| = \delta\}$ tending to 0 as δ does, while the other satisfies:

$$\begin{aligned} \int_{\{\mathbf{y}\,:\|\mathbf{y}\| = R\}} \left| u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right| d_S \mathbf{y} &\le \int_{S^{d-1}} \left(\frac{C_1 R \log R}{R^{d+\epsilon}} + \frac{C_2 R^2 \log R}{R^{d+1+\epsilon}} \right) R^{d-1}\, d\Omega \\ &= \frac{(C_1 + C_2) \log R \omega_d}{R^\epsilon}, \end{aligned}$$

which approaches 0 as R tends to ∞. Note here that

$$\begin{aligned} \left| \frac{\partial v}{\partial n}(\mathbf{y}) \right| &= \left| \sum_{i=1}^d \frac{y_i}{R} \frac{\partial v}{\partial y_i}(\mathbf{y}) \right| = \left| \frac{1}{R} \int_{S^{d-1}} \mathbf{y} \cdot \mathbf{e} \beta'(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))\, d\mathbf{e} \right| \le \int_{S^{d-1}} |\beta'(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))|\, d\mathbf{e} \\ &\le |\beta'(\| \mathbf{x} - \mathbf{y} \|) \omega_d| \le C_1 R \log R \end{aligned}$$

and

$$|v(\mathbf{y})| \le \int_{S^{d-1}} |\beta(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))|\, d\mathbf{e} \le C_2 R^2 \log R$$

for $\| \mathbf{y} \| = R$ large and suitable constants C_1 and C_2. So (7.6) holds.

From Lemma 6.1, and the identities $\int_{S^{d-1}} g(-\mathbf{e})\, d\mathbf{e} = \int_{S^{d-1}} g(\mathbf{e})\, d\mathbf{e}$ and $\beta'(-t) = -\beta'(t)$, we obtain:

$$\begin{aligned} &\int_{S^{d-1}} \beta(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))\, d\mathbf{e} \\ &= \int_{S^{d-1}} \left(\int_0^\infty \beta'(t) \vartheta(\mathbf{e} \cdot (\mathbf{x} - \mathbf{y}) - t)\, dt - \int_{-\infty}^0 \beta'(t) \vartheta(t - \mathbf{e} \cdot (\mathbf{x} - \mathbf{y}))\, dt \right) d\mathbf{e} \\ &= \int_{S^{d-1}} \left(\int_{-\infty}^{-\mathbf{e} \cdot \mathbf{y}} \beta'(-\mathbf{e} \cdot \mathbf{y} - b) \vartheta(\mathbf{e} \cdot \mathbf{x} + b)\, db - \int_{-\infty}^{\mathbf{e} \cdot \mathbf{y}} \beta'(b - \mathbf{e} \cdot \mathbf{y}) \vartheta(b - \mathbf{e} \cdot \mathbf{x})\, db \right) d\mathbf{e} \\ &= 2 \int_{S^{d-1}} \int_{\mathcal{R}} \beta'(-\mathbf{e} \cdot \mathbf{y} - b) \vartheta(-\mathbf{e} \cdot \mathbf{y} - b) \vartheta(\mathbf{e} \cdot \mathbf{x} + b)\, db\, d\mathbf{e}. \end{aligned}$$

Substituting the last expression into (7.6) and rearranging terms, we have:

$$f(\mathbf{x}) = \frac{2c_d}{\omega_d} \int_{S^{d-1}} \int_{\mathcal{R}} \left(\int_{\mathcal{R}^d} \triangle^{\frac{d+2}{2}} f(\mathbf{y}) \beta'(-\mathbf{e} \cdot \mathbf{y} - b) \vartheta(-\mathbf{e} \cdot \mathbf{y} - b)\, d\mathbf{y} \right) \vartheta(\mathbf{e} \cdot \mathbf{x} + b)\, db\, d\mathbf{e}.$$

Hence,

$$f(\mathbf{x}) = \int_{S^{d-1}\times\mathcal{R}} \hat{w}_f(\mathbf{e}, b)\vartheta(\mathbf{x}\cdot\mathbf{e}+b)\, d\mathbf{e}\, db,$$

where

$$\hat{w}_f(\mathbf{e}, b) = \frac{2c_d}{\omega_d}\int_{H^-_{\mathbf{e},b}} \triangle^{\frac{d+2}{2}} f(\mathbf{y})\alpha(\mathbf{e}\cdot\mathbf{y}+b)\, d\mathbf{y}, \tag{7.7}$$

since $\alpha(t) = \beta'(-t)$.

As before, change in the order of integration is justified by Fubini's Theorem. Indeed

$$\begin{aligned}
&\int_{\mathcal{R}^d}\int_{S^{d-1}}\int_{\mathcal{R}} \left|\triangle^{\frac{d+2}{2}} f(\mathbf{y})\right| |\alpha(\mathbf{e}\cdot\mathbf{y}+b)||\vartheta(-\mathbf{e}\cdot\mathbf{y}-b)\vartheta(\mathbf{e}\cdot\mathbf{x}+b)|\, db\, d\mathbf{e}\, d\mathbf{y}\\
&\quad\le \int_{\mathcal{R}^d}\int_{S^{d-1}} \left|\triangle^{\frac{d+2}{2}} f(\mathbf{y})\right| \max\{1, |\alpha(|\mathbf{e}\cdot(\mathbf{y}-\mathbf{x})|)|\}|\mathbf{e}\cdot(\mathbf{y}-\mathbf{x})|\, d\mathbf{e}\, d\mathbf{y}\\
&\quad\le \omega_d\int_{\mathcal{R}^d} \left|\triangle^{\frac{d+2}{2}} f(\mathbf{y})\right| \max\{1, |\alpha(\|\,\mathbf{y}-\mathbf{x}\,\|)|\}\,\|\,\mathbf{y}-\mathbf{x}\,\|\ d\mathbf{y},
\end{aligned}$$

which follows from the inequality $|\alpha(t)| \le \max\{1, |\alpha(s)|\}$ for $|t| \le |s|$. Since

$$\lim_{\|\mathbf{y}\|\longrightarrow\infty} \frac{\|\,\mathbf{x}-\mathbf{y}\,\|\,|\alpha(\|\,\mathbf{x}-\mathbf{y}\,\|)|}{\|\,\mathbf{y}\,\|\,|\alpha(\|\,\mathbf{y}\,\|)|} = 1,$$

we take R large and boundedness follows from:

$$\begin{aligned}
\int_{\{\mathbf{y}:\|\mathbf{y}\|\ge R\}} \left|\triangle^{\frac{d+2}{2}} f(\mathbf{y})\right|\,\|\,\mathbf{y}\,\|\,|\alpha(\|\,\mathbf{y}\,\|)|\, d\mathbf{y} &\le \int_{\|\mathbf{y}\|\ge R} \frac{Cr^2(\log r)r^{d-1}\, dr\, d\Omega}{r^{d+2+\epsilon}}\\
&= C\omega_d\int_R^\infty \frac{\log r\, dr}{r^{1+\epsilon}}\\
&< \infty.
\end{aligned}$$

In particular $\hat{w}_f(\mathbf{e}, b)\vartheta(\mathbf{e}\cdot\mathbf{x}+b)$ is integrable on $S^{d-1}\times\mathcal{R}$.

Properties of $\hat{w}_f$. Equation (7.7) gives us a weight function $\hat{w}_f$ for an integral formula. However, to obtain the weight function w_f in the theorem, we first need two properties of $\hat{w}_f$, namely:

(P1) For $b_0 \in \mathcal{R}$ there exist $\eta > 0$ and $M > 0$ such that $|\hat{w}_f(\mathbf{e}, b)| \le M/(b^2+1)^{(1+\eta)/2}$ for all $\mathbf{e} \in S^{d-1}$ and all $b \ge b_0$; and

(P2) $\lim_{K\longrightarrow\infty}\int_{-K}^{\infty} \hat{w}_f(\mathbf{e}, b)\, db = 0$ uniformly for $\mathbf{e} \in S^{d-1}$.

To establish (P1), fix $\mathbf{e}$, choose rectangular coordinates $\mathbf{y} = (y_1, \mathbf{y}^\perp)$ with $y_1 = \mathbf{e}\cdot\mathbf{y}$, and set $\rho = \|\,\mathbf{y}^\perp\,\|$. Then, arguing as before, we have:

$$\begin{aligned}
\frac{\omega_d}{2c_d}|\hat{w}_f(\mathbf{e}, b)| &\le \int_{H^-_{\mathbf{e},b}} \left|\triangle^{\frac{d+2}{2}} f(\mathbf{y})\right| |\alpha(\mathbf{e}\cdot\mathbf{y}+b)|\, d\mathbf{y}\\
&\le \int_{-\infty}^{-b}\int_{\mathbf{y}^\perp\in\mathcal{R}^{d-1}} \frac{C|\alpha(y_1+b)|\, d\mathbf{y}^\perp\, dy_1}{(\|\,\mathbf{y}\,\|^2+1)^{(d+2+\epsilon)/2}}\\
&= \int_{-\infty}^{-b}\int_{[0,\infty)\times S^{d-2}} \frac{C|\alpha(y_1+b)|\rho^{d-2}\, d\rho\, d\Omega\, dy_1}{(y_1^2+\rho^2+1)^{(d+2+\epsilon)/2}}\\
&= C\omega_{d-1}\int_{y_1=-\infty}^{-b}\int_{\theta=0}^{\frac{\pi}{2}} \frac{|\alpha(y_1+b)|\sin^{d-2}\theta\cos^{2+\epsilon}\theta\, d\theta\, dy_1}{(y_1^2+1)^{(3+\epsilon)/2}} \le
\end{aligned}$$

$$\leq C\omega_{d-1}\frac{\pi}{2}\int_{y_1=-\infty}^{-b}\frac{|\alpha(y_1+b)|\,dy_1}{(y_1^2+1)^{(3+\epsilon)/2}}$$
$$= C\omega_{d-1}\frac{\pi}{2}\int_{0}^{\infty}\frac{|\alpha(t)|\,dt}{((t+b)^2+1)^{(3+\epsilon)/2}}.$$

The last integral is a continuous function of b defined for all values of b. Noting that $|\alpha(t)| \leq 1$ for $|t| \leq e$ and $|\alpha(t)| \leq \frac{t^{1+\delta}}{\delta(1+\delta)}$ for $t \geq e$ and any $\delta > 0$, we see that when $b \geq 0$ and $\epsilon > \delta > 0$:

$$\left(\int_0^e + \int_e^\infty\right)\frac{|\alpha(t)|\,dt}{((t+b)^2+1)^{(3+\epsilon)/2}} \leq \frac{e}{(b^2+1)^{(3+\epsilon)/2}} + \int_e^\infty \frac{(t+b)^{1+\delta}\,dt}{\delta(1+\delta)(t+b)^{3+\epsilon}}$$
$$\leq \frac{e}{(b^2+1)^{(3+\epsilon)/2}} + \frac{1}{\delta(1+\delta)(1+\epsilon-\delta)(e+b)^{1+\epsilon-\delta}}$$
$$\leq \frac{e+\frac{1}{\delta(1+\delta)(1+\epsilon-\delta)}}{(b^2+1)^{(1+\epsilon-\delta)/2}}.$$

Hence, (P1) holds for $b \geq 0$ and $\eta = \epsilon - \delta$. If $b_0 < 0$, continuity on the interval $[b_0, 0]$ allows us to draw the same conclusion for $b \geq b_0$ (with the constant M replaced by a larger constant that depends on b_0).

To prove (P2), note first that (P1) implies that for fixed $\mathbf{e}$ the mapping $b \longmapsto \hat{w}_f(\mathbf{e}, b)$ is integrable on intervals of the form $[-K, \infty)$. Furthermore, Fubini's Theorem in combination with the argument for (P1) allows us to change the order of integration. Since all partials with respect to variables other than y_1 can be integrated out and evaluated at infinity (where the antiderivatives vanish), we find that

$$\frac{\omega_d}{2c_d}\int_{-K}^\infty \hat{w}_f(\mathbf{e}, b)\,db = \int_{-K}^\infty \int_{H_{\mathbf{e},b}^-} \triangle^{\frac{d+2}{2}} f(\mathbf{y})\alpha(\mathbf{e}\cdot\mathbf{y}+b)\,d\mathbf{y}\,db$$
$$= \int_{-K}^\infty \int_{y_1=-\infty}^{-b}\int_{\mathbf{y}^\perp\in\mathcal{R}^{d-1}} \triangle^{\frac{d+2}{2}} f(\mathbf{y})\alpha(y_1+b)\,d\mathbf{y}^\perp\,dy_1\,db$$
$$= \int_{-K}^\infty \int_{y_1=-\infty}^{-b}\int_{\mathbf{y}^\perp\in\mathcal{R}^{d-1}} \left(\frac{\partial}{\partial y_1}\right)^{d+2} f(\mathbf{y})\alpha(y_1+b)\,d\mathbf{y}^\perp\,dy_1\,db$$
$$= \int_{\mathbf{y}^\perp\in\mathcal{R}^{d-1}} I(\mathbf{y}^\perp)\,d\mathbf{y}^\perp,$$

where

$$I(\mathbf{y}^\perp) = \int_{-K}^\infty\int_{-\infty}^{-b}\left(\frac{\partial}{\partial y_1}\right)^{d+2} f(\mathbf{y})\alpha(y_1+b)\,dy_1\,db$$
$$= \int_{-\infty}^{K}\int_{-K}^{-y_1}\left(\frac{\partial}{\partial y_1}\right)^{d+2} f(\mathbf{y})\alpha(y_1+b)\,db\,dy_1$$
$$= \int_{-\infty}^{K}\int_{-K}^{-y_1}\left(\frac{\partial}{\partial y_1}\right)^{d+2} f(\mathbf{y})\beta'(-y_1-b)\,db\,dy_1$$
$$= \int_{-\infty}^{K}\left(\frac{\partial}{\partial y_1}\right)^{d+2} f(\mathbf{y})\beta(K-y_1)\,dy_1$$
$$= \int_{-\infty}^{K}\left(\frac{\partial}{\partial y_1}\right)^{d+1} f(\mathbf{y})\alpha(y_1-K)\,dy_1$$
$$= \int_{-\infty}^{K^-}\left(\frac{\partial}{\partial y_1}\right)^{d} f(\mathbf{y})\log|y_1-K|\,dy_1.$$

Here we have used integration by parts, properties of β and α, and the decay condition satisfied by f. The last integral decomposes further as:

$$
\begin{aligned}
&\left(\int_{-\infty}^{K-1}+\int_{K-1}^{K^-}\right)\left(\frac{\partial}{\partial y_1}\right)^d f(\mathbf{y})\log|y_1-K|\,dy_1\\
&= \log|y_1-K|\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(\mathbf{y})\Bigg|_{y_1=-\infty}^{K-1} + \int_{-\infty}^{K-1}\frac{(\frac{\partial}{\partial y_1})^{d-1}f(\mathbf{y})\,dy_1}{K-y_1}\\
&\quad + (y_1-K)\log|y_1-K|\frac{(\frac{\partial}{\partial y_1})^{d-1}f(\mathbf{y})-(\frac{\partial}{\partial y_1})^{d-1}f(K,\mathbf{y}^\perp)}{y_1-K}\Bigg|_{K-1}^{K^-}\\
&\quad + \int_{K-1}^{K^-}\left(\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(\mathbf{y})-\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(K,\mathbf{y}^\perp)\right)\left(\frac{1}{K-y_1}\right)dy_1\\
&= \int_{-\infty}^{K-1}\frac{\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(\mathbf{y})\,dy_1}{K-y_1} + \int_{K-1}^{K^-}\left(\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(\mathbf{y})-\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(K,\mathbf{y}^\perp)\right)\left(\frac{1}{K-y_1}\right)dy_1\\
&=: I_1(\mathbf{y}^\perp)+I_2(\mathbf{y}^\perp).
\end{aligned}
$$

We now integrate with respect to $\mathbf{y}^\perp$. For $K>1$ by the Mean Value Theorem:

$$
\begin{aligned}
&\left|\int_{\mathcal{R}^{d-1}} I_2(\mathbf{y}^\perp)\,d\mathbf{y}^\perp\right|\\
&\quad = \left|\int_{K-1}^{K^-}\int_{\mathcal{R}^{d-1}}\left(\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(\mathbf{y})-\left(\frac{\partial}{\partial y_1}\right)^{d-1} f(K,\mathbf{y}^\perp)\right)\left(\frac{1}{y_1-K}\right)dy_1\,d\mathbf{y}^\perp\right|\\
&\quad \le \int_{\mathcal{R}^{d-1}}\max_{K-1\le y_1\le K}\left|\left(\frac{\partial}{\partial y_1}\right)^d f(\mathbf{y})\right|d\mathbf{y}^\perp\\
&\quad \le \int_{[0,\infty)\times S^{d-2}}\frac{C\rho^{d-2}\,d\Omega\,d\rho}{((K-1)^2+\rho^2+1)^{(d+\epsilon)/2}}\\
&\quad = C\omega_{d-1}\int_0^{\pi/2}\frac{1}{((K-1)^2+1)^{(1+\epsilon)/2}}\sin^{d-2}\theta\cos^{\epsilon}\theta\,d\theta\\
&\quad \le C\omega_{d-1}\frac{\pi}{2((K-1)^2+1)^{(1+\epsilon)/2}},
\end{aligned}
$$

where we have used the substitution $\rho=\left(\sqrt{(K-1)^2+1}\right)\tan\theta$. As K tends to infinity, this integral tends to 0.

Likewise,

$$
\begin{aligned}
\left|\int_{\mathcal{R}^{d-1}} I_1(\mathbf{y}^\perp)\,d\mathbf{y}^\perp\right| &= \left|\int_{-\infty}^{K-1}\int_{\mathcal{R}^{d-1}}\frac{(\frac{\partial}{\partial y_1})^{d-1}f(\mathbf{y})\,d\mathbf{y}^\perp\,dy_1}{y_1-K}\right|\\
&\le \int_{-\infty}^{K-1}\int_{[0,\infty)\times S^{d-2}}\frac{C\rho^{d-2}\,d\Omega\,d\rho\,dy_1}{|y_1-K|(y_1^2+\rho^2+1)^{(d-1+\epsilon)/2}}\\
&= C\omega_{d-1}\int_{-\infty}^{K-1}\int_0^{\frac{\pi}{2}}\frac{\sin^{d-2}\theta\,dy_1\,d\theta}{|y_1-K|(y_1^2+1)^{\epsilon/2}\cos^{1-\epsilon}\theta}\\
&\le \frac{C\omega_{d-1}\pi}{2\epsilon}\int_{-\infty}^{K-1}\frac{dy_1}{|y_1-K|(y_1^2+1)^{\epsilon/2}},
\end{aligned}
$$

which uses the inequality $\cos^{1-\epsilon}\theta\ge\left(1-\frac{2}{\pi}\theta\right)^{1-\epsilon}$ for $\epsilon\le 1$ and $0\le\theta\le\frac{\pi}{2}$. Ignoring the scale factors in front and making the substitution $u=K-y_1$, we rewrite the last integral as:

$$\begin{aligned}
&\int_1^\infty \frac{du}{u((K-u)^2+1)^{\epsilon/2}} \\
&\quad = \left(\int_1^{K-1} + \int_{K-1}^{K+1} + \int_{K+1}^\infty\right) \frac{du}{u((K-u)^2+1)^{\epsilon/2}} \\
&\quad \le \int_1^{K-1} \frac{du}{u(K-u)^\epsilon} + \int_{K-1}^{K+1} \frac{du}{u} + \int_1^\infty \frac{dv}{(v+K)(v^2+1)^{\epsilon/2}} \\
&\quad \le \frac{(K-1)^{1-\epsilon}}{K} \int_1^{K-1} \left(\frac{1}{u} + \frac{1}{K-u}\right) du + \log\frac{K+1}{K-1} + \int_1^\infty \frac{dv}{(v+K)(v^2+1)^{\epsilon/2}} \\
&\quad = \frac{2(K-1)^{1-\epsilon}\log(K-1)}{K} + \log\frac{K+1}{K-1} + \int_1^\infty \frac{dv}{(v+K)(v^2+1)^{\epsilon/2}}.
\end{aligned}$$

As K tends to ∞, the first two terms tend to 0, while the integral is dominated by $\int_1^\infty \frac{dv}{v^{1+\epsilon}} = \frac{1}{\epsilon}$ and so also tends to 0 by Lebesgue's Dominated Convergence Theorem. Thus (P2) is established.

Replacing $\hat{w}_f$ by w_f. Since $a_d = \frac{c_d}{\omega_d}$ for d even, we can express w_f in terms of $\hat{w}_f$, using $\int_{\mathcal{R}^d} = \int_{H^+_{\mathbf{e},b}} + \int_{H^-_{\mathbf{e},b}}$ and the oddness of α, as:

$$w_f(\mathbf{e}, b) = \frac{1}{2}(\hat{w}_f(\mathbf{e}, b) - \hat{w}_f(-\mathbf{e}, -b)).$$

Then

$$\begin{aligned}
&\int_{S^{d-1}\times(-\infty,K]} w_f(\mathbf{e}, b)\theta(\mathbf{e}\cdot\mathbf{x}+b)\, d\mathbf{e}\, db \\
&\quad = \int_{S^{d-1}\times(-\infty,K]} \hat{w}_f(\mathbf{e}, b)\theta(\mathbf{e}\cdot\mathbf{x}+b)\, d\mathbf{e}\, db - \int_{S^{d-1}\times(-\infty,K]} \frac{1}{2}(\hat{w}_f(\mathbf{e}, b) + \hat{w}_f(-\mathbf{e}, -b))\theta(\mathbf{e}\cdot\mathbf{x}+b)\, d\mathbf{e}\, db.
\end{aligned}$$

The first integral in the last expression tends to $f(\mathbf{x})$ as K tends to ∞ by (7.7). So it suffices to show that the second integral tends to 0 as K tends to ∞. Indeed, for $K > \| \mathbf{x} \|$,

$$\begin{aligned}
&\int_{S^{d-1}\times(-\infty,K]} (\hat{w}_f(\mathbf{e}, b) + \hat{w}_f(-\mathbf{e}, -b))\theta(\mathbf{e}\cdot\mathbf{x}+b)\, d\mathbf{e}\, db \\
&\quad = \int_{S^{d-1}\times(-\infty,K]} \hat{w}_f(\mathbf{e}, b)\theta(\mathbf{e}\cdot\mathbf{x}+b)\, d\mathbf{e}\, db + \int_{S^{d-1}\times[-K,\infty)} \hat{w}_f(\mathbf{e}, b)\theta(-\mathbf{e}\cdot\mathbf{x}-b)\, d\mathbf{e}\, db \\
&\quad = \int_{S^{d-1}} \int_{-\mathbf{e}\cdot\mathbf{x}}^{\max\{K,-\mathbf{e}\cdot\mathbf{x}\}} \hat{w}_f(\mathbf{e}, b)\, db\, d\mathbf{e} + \int_{S^{d-1}} \int_{\min\{-K,-\mathbf{e}\cdot\mathbf{x}\}}^{-\mathbf{e}\cdot\mathbf{x}} \hat{w}_f(\mathbf{e}, b)\, db\, d\mathbf{e} \\
&\quad = \int_{S^{d-1}} \int_{-K}^{K} \hat{w}_f(\mathbf{e}, b)\, db\, d\mathbf{e}.
\end{aligned}$$

However, $\int_{-K}^{K} \hat{w}_f(\mathbf{e}, b)\, db$ tends to 0 uniformly in $\mathbf{e}$ as K tends to ∞ since by (P1) $\int_K^\infty$ tends to 0 and by (P2) $\int_{-K}^{\infty}$ tends to 0. This completes the proof. □

8 Alternative representations

The formulas for w_f in the Representation Theorem (Theorem 4.2) can be written in several alternative forms, some of which have appeared in the literature under stronger hypotheses.

Kůrková, Kainen and Kreinovich [19] used distributional techniques from Courant and Hilbert [5] to show that if f is a compactly supported function on $\mathcal{R}^d$ with continuous d-th order partials, and d is odd, then f can be represented as in (4.1), where w_f is as in (8.3) below.

Ito [13] and Carroll and Dickinson [4] treated both the odd and the even case, basing their work on Helgason's book on the Radon Transform [11], and obtained a representation for C^∞ functions of rapid descent (Ito) and C^∞ functions of compact support (Carroll and Dickinson).

The connection of previous work to the two propositions below is discussed at the end of this section.

In the following proposition we make use of principal value integration (cf. Zemanian [20, p. 18]) and directional derivatives. Thus p.v. $\int_{\mathcal{R}^d} \frac{\phi(\mathbf{y})\,d\mathbf{y}}{\mathbf{e}\cdot\mathbf{y}+b} := \lim_{\delta\longrightarrow 0+} \int_{\{\mathbf{y}\,:\,|\mathbf{e}\cdot\mathbf{y}+b|\geq\delta\}} \frac{\phi(\mathbf{y})\,d\mathbf{y}}{\mathbf{e}\cdot\mathbf{y}+b}$ provided the latter exists; and $D_{\mathbf{e}}^{(k)}$ refers to the k-th order directional derivative in the direction $\mathbf{e}$.

Proposition 8.1 *Let f be of controlled decay. For d odd*

$$w_f(\mathbf{e},b) = a_d \int_{H^-_{\mathbf{e},b}} \triangle^{\frac{d+1}{2}} f(\mathbf{y})\,d\mathbf{y} \tag{8.1}$$

$$= a_d \int_{H^-_{\mathbf{e},b}} D_{\mathbf{e}}^{(d+1)} f(\mathbf{y})\,d\mathbf{y} \tag{8.2}$$

$$= a_d \int_{H_{\mathbf{e},b}} D_{\mathbf{e}}^{(d)} f(\mathbf{y})\,d_H\mathbf{y}. \tag{8.3}$$

For d even,

$$w_f(\mathbf{e},b) = a_d \int_{\mathcal{R}^d} \triangle^{\frac{d+2}{2}} f(\mathbf{y})\alpha(\mathbf{e}\cdot\mathbf{y}+b)\,d\mathbf{y} \tag{8.4}$$

$$= a_d \int_{\mathcal{R}^d} D_{\mathbf{e}}^{(d+2)} f(\mathbf{y})\alpha(\mathbf{e}\cdot\mathbf{y}+b)\,d\mathbf{y} \tag{8.5}$$

$$= a_d \int_{\mathcal{R}^d} D_{\mathbf{e}}^{(d+1)} f(\mathbf{y}) \log|\mathbf{e}\cdot\mathbf{y}+b|\,d\mathbf{y} \tag{8.6}$$

$$= -a_d\, p.v. \int_{\mathcal{R}^d} \frac{D_{\mathbf{e}}^{(d)} f(\mathbf{y})}{\mathbf{e}\cdot\mathbf{y}+b}\,d\mathbf{y}. \tag{8.7}$$

If f is of controlled decay and also satisfies

$$ord\, \partial^\alpha f > d-1 \quad for \quad 0 \leq |\alpha| \leq d-2, \tag{8.8}$$

then for d odd

$$w_f(\mathbf{e},b) = -a_d \left(\frac{\partial}{\partial a}\right)^d \left(\int_{H_{\mathbf{e},a}} f(\mathbf{y})\,d_H\mathbf{y}\right)\Bigg|_{a=b} \tag{8.9}$$

and for d even

$$w_f(\mathbf{e},b) = -a_d \left(\frac{\partial}{\partial a}\right)^d \left(p.v. \int_{\mathcal{R}^d} \frac{f(\mathbf{y})}{\mathbf{e}\cdot\mathbf{y}+a}\,d\mathbf{y}\right)\Bigg|_{a=b}. \tag{8.10}$$

Proof. **The odd case.** In the case $d = 1$ Equations (8.1)–(8.3), and (8.9) are trivial and $w_f(\mathbf{e},b)$ agrees with Proposition 5.1. Suppose $d \geq 3$. The integral in (8.1), from Theorem 4.2,

$$\int_{H^-_{\mathbf{e},b}} \triangle^{\frac{d+1}{2}} f(\mathbf{y})\,d\mathbf{y}$$

is an integral over the half-space where $\mathbf{y}\cdot\mathbf{e}+b \leq 0$. If we adopt a rectangular coordinate system in which $y_1 = \mathbf{y}\cdot\mathbf{e}$, the iterated Laplacian $\triangle^{\frac{d+1}{2}} f(\mathbf{y})$ retains its usual form with $\triangle = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_d^2}$. The integrand consists of a sum of partial derivatives of f of order $d+1$ all but one of which can be expressed in the form $\frac{\partial u}{\partial y_i}$

for some $i \neq 1$ and some function u, itself a partial derivative of f of order d, by interchanging the order of differentiation. Integrating each such term, but integrating first with respect to y_i, we find that

$$\begin{aligned}\int_{H^-_{\mathbf{e},b}} \frac{\partial u}{\partial y_i}(\mathbf{y})\, d\mathbf{y} &= \int_{y_1 \le -b, (y_2,\dots,y_{i-1},y_{i+1},\dots,y_d)\in\mathcal{R}^{d-2}} \left(\int_{y_i=-\infty}^{\infty} \frac{\partial u}{\partial y_i}(\mathbf{y}) dy_i\right) dy_1 \dots dy_{i-1} dy_{i+1} \dots dy_d \\ &= \int_{y_1 \le -b, (y_2,\dots,y_{i-1},y_{i+1},\dots,y_d)\in\mathcal{R}^{d-2}} u(\mathbf{y})|_{yi=-\infty}^{\infty}\, dy_1 \dots dy_{i-1} dy_{i+1} \dots dy_d \\ &= 0\end{aligned}$$

since $\lim_{y_i \longrightarrow \infty} u(\mathbf{y}) = \lim_{\|\mathbf{y}\| \longrightarrow \infty} u(\mathbf{y}) = 0$. Use of Fubini's Theorem is justified since $|\frac{\partial u}{\partial y_i}|$, a derivative of order $d+1$ dominated at infinity by $\frac{1}{\|\mathbf{y}\|^{d+1+\epsilon}}$, is integrable over $\mathcal{R}^d$ and over $H^-_{\mathbf{e},b}$.

Accordingly the integral of interest reduces to:

$$\int_{H^-_{\mathbf{e},b}} \left(\frac{\partial}{\partial y_1}\right)^{d+1} f(\mathbf{y})\, d\mathbf{y}.$$

Then we integrate with respect to y_1 first of all, obtaining:

$$\begin{aligned}\int_{\mathcal{R}^{d-1}} \left(\frac{\partial}{\partial y_1}\right)^{d} f(\mathbf{y})\Bigg|_{y_1=-\infty}^{y_1=-b} dy_2 \dots dy_d &= \int_{\mathcal{R}^{d-1}} \left(\frac{\partial}{\partial y_1}\right)^{d} f(-b, \mathbf{y}^\perp)\, d\mathbf{y}^\perp \\ &= -\left(\frac{\partial}{\partial a}\right)^{d} \left(\int_{\mathcal{R}^{d-1}} f(-a, \mathbf{y}^\perp)\, d\mathbf{y}^\perp\right)\Bigg|_{a=b},\end{aligned}$$

where shifting the partial derivatives outside the integral is justified below. The above expressions are identical, except for the omitted factor a_d, with (8.2), (8.3), and (8.9) above since $d\mathbf{y}^\perp = d_H\mathbf{y}$, and $\frac{\partial}{\partial y_1}$ and its iterates are directional derivatives in the direction of $\mathbf{e}$, i.e., normal to the hyperplane $H_{\mathbf{e},b}$.

Shifting the partials is permitted if we show that

$$\int_{\mathcal{R}^{d-1}} \frac{\partial v}{\partial y_i}(-b, \mathbf{y}^\perp)\, d\mathbf{y}^\perp = -\left(\frac{\partial}{\partial a}\right) \left(\int_{\mathcal{R}^{d-1}} v(-a, \mathbf{y}^\perp)\, d\mathbf{y}^\perp\right)\Bigg|_{a=b}$$

for any real number b where $v = \left(\frac{\partial}{\partial y_1}\right)^i f$ for $i = 0, \dots, d-1$. By the definition of a derivative, it suffices to show that

$$\begin{aligned}&\int_{\mathcal{R}^{d-1}} \left(\frac{v(-b-h, \mathbf{y}^\perp) - v(-b, \mathbf{y}^\perp)}{h} + \frac{\partial v}{\partial y_1}(-b, \mathbf{y}^\perp)\right) d\mathbf{y}^\perp \\ &\quad = \int_{\mathcal{R}^{d-1}} \left(\int_{s=0}^{1} \left(\frac{\partial v}{\partial y_1}(-b, \mathbf{y}^\perp) - \frac{\partial v}{\partial y_1}(-b-sh, \mathbf{y}^\perp)\right) ds\right) d\mathbf{y}^\perp\end{aligned}$$

tends to 0 as h approaches 0.

The controlled decay property and (8.8) imply that $\lim_{\|\mathbf{y}\| \longrightarrow \infty} \frac{\partial v}{\partial y_1}(\mathbf{y})\|\mathbf{y}\|^{d-1+\epsilon} = 0$ for all v considered. The last integral over $\mathcal{R}^{d-1}$ decomposes into two parts: one for $\|\mathbf{y}^\perp\|$ large, say, larger than R, where the inner integral is dominated by $\frac{2\epsilon_0}{\|\mathbf{y}^\perp\|^{d-1+\epsilon}}$ and integration with respect to $d\mathbf{y}^\perp = \rho^{d-2} d\rho\, d\Omega$ with $\rho = \|\mathbf{y}^\perp\|$ yields an answer dominated by $\frac{2\epsilon_0 \omega_{d-1}}{\epsilon R^\epsilon}$ (which can be made arbitrarily small by letting R tend to infinity); while the other part is over the compact set $C = \{\mathbf{y}^\perp : \|\mathbf{y}^\perp\| \le R\}$ where uniform continuity of $\frac{\partial v}{\partial y_1}(-b, \mathbf{y}^\perp)$ guarantees that for h sufficiently close to 0 the integrand and the integral over C are arbitrarily small. Thus the entire integral tends to 0 as h does, to complete the argument.

The even case. The representation in (8.4) derives from Theorem 4.2. In a rectangular coordinate system in which $y_1 = \mathbf{y} \cdot \mathbf{e}$, just as in the odd case, by dropping terms that have partial derivatives with respect to variables other than y_1, we arrive at the following equations, with $D_{\mathbf{e}} = \frac{\partial}{\partial y_1}$ and p.v. $\int_{\mathcal{R}^d} \phi(\mathbf{y})\, d\mathbf{y} := \lim_{\delta \longrightarrow 0} \int_{\{\mathbf{y}\,:\,|y_1+b|\geq\delta\}} \phi(\mathbf{y})\, d\mathbf{y}$:

$$\begin{aligned}\frac{w_f(\mathbf{e},b)}{a_d} = \int_{\mathcal{R}^d} \left(\frac{\partial}{\partial y_1}\right)^{d+2} f(\mathbf{y})\alpha(y_1+b)\, d\mathbf{y} &\overset{(i)}{=} \int_{\mathcal{R}^d} \left(\frac{\partial}{\partial y_1}\right)^{d+1} f(\mathbf{y}) \log|y_1+b|\, d\mathbf{y} \\ &\overset{(ii)}{=} -\text{p.v.} \int_{\mathcal{R}^d} \left(\frac{\partial}{\partial y_1}\right)^{d} f(\mathbf{y}) \frac{1}{y_1+b}\, d\mathbf{y} \\ &\overset{(iii)}{=} -\left(\frac{\partial}{\partial a}\right)^{d} \left(\text{p.v.} \int_{\mathcal{R}^d} \frac{f(\mathbf{y})}{y_1+a}\, d\mathbf{y}\right)\Bigg|_{a=b}.\end{aligned}$$

The left side of (i) gives (8.5). So if we establish (i), (ii), and (iii), then (8.6), (8.7), and (8.10) follow.

An integration by parts with respect to the variable y_1 yields (i). As the function $\log|y_1+b| = -\alpha'(y_1+b)$ has a singularity at $y_1 = -b$, Lebesgue integration is required.

Another integration by parts leads to (ii). Indeed, for $\delta > 0$

$$\begin{aligned}&\int_{\{\mathbf{y}\,:\,|y_1+b|\geq\delta\}} \left(\frac{\partial}{\partial y_1}\right)^{d+1} f(\mathbf{y}) \log|y_1+b|\, d\mathbf{y} \\ &\quad = -\int_{\mathcal{R}^{d-1}} \left(\frac{\partial}{\partial y_1}\right)^{d} f(-b+\delta, \mathbf{y}^{\perp}) \log\delta\, d\mathbf{y}^{\perp} - \int_{\{\mathbf{y}\,:\,y_1+b\geq\delta\}} \left(\frac{\partial}{\partial y_1}\right)^{d} f(\mathbf{y}) \frac{1}{y_1+b}\, d\mathbf{y} \\ &\qquad + \int_{\mathcal{R}^{d-1}} \left(\frac{\partial}{\partial y_1}\right)^{d} f(-b-\delta, \mathbf{y}^{\perp}) \log\delta\, d\mathbf{y}^{\perp} - \int_{\{\mathbf{y}\,:\,y_1+b\leq-\delta\}} \left(\frac{\partial}{\partial y_1}\right)^{d} f(\mathbf{y}) \frac{1}{y_1+b}\, d\mathbf{y} \\ &\quad = \delta\log\delta \left(-\int_{\mathcal{R}^{d-1}} \int_{t=-1}^{1} \left(\frac{\partial}{\partial y_1}\right)^{d+1} f(-b+t\delta, \mathbf{y}^{\perp})\, dt\, d\mathbf{y}^{\perp}\right) \\ &\qquad - \int_{\{\mathbf{y}\,:\,|y_1+b|\geq\delta\}} \left(\frac{\partial}{\partial y_1}\right)^{d} f(\mathbf{y}) \frac{1}{y_1+b}\, d\mathbf{y}.\end{aligned}$$

The coefficient of $\delta\log\delta$ is a well-defined finite integral, and as δ tends to $0+$, $\delta\log\delta$ tends to 0.

To show (iii), we must establish the following:

$$\text{p.v.} \int_{\mathcal{R}^d} \frac{\frac{\partial v}{\partial y_1}}{y_1+b}\, d\mathbf{y} = -\left(\frac{\partial}{\partial a}\right) \left(\text{p.v.} \int_{\mathcal{R}^d} \frac{v(\mathbf{y})}{y_1+a}\, d\mathbf{y}\right)\Bigg|_{a=b}$$

for all real numbers b and $v = \left(\frac{\partial}{\partial y_1}\right)^{i} f$ for $i = 0, \ldots, d-1$. Consider now the differential quotient

$$\begin{aligned}&\frac{1}{h}\left(\text{p.v.} \int_{\mathcal{R}^d} \frac{v(\mathbf{y})\, d\mathbf{y}}{y_1+b+h} - \text{p.v.} \int_{\mathcal{R}^d} \frac{v(\mathbf{y})\, d\mathbf{y}}{y_1+b}\right) \\ &\quad = \text{p.v.} \left(\int_{\mathcal{R}^d} \frac{1}{y_1+b} \left(\frac{v(y_1-h, \mathbf{y}^{\perp}) - v(\mathbf{y})}{h} + \frac{\partial v}{\partial y_1}(\mathbf{y})\right) d\mathbf{y}\right) \\ &\quad = \text{p.v.} \int_{\mathcal{R}^d} \frac{1}{y_1+b} H(\mathbf{y},h)\, d\mathbf{y} \\ &\quad = \int_{\{\mathbf{y}\in\mathcal{R}^d\,:\,|y_1+h|\geq 1\}} \frac{1}{y_1+b} H(\mathbf{y},h)\, d\mathbf{y} + \text{p.v.} \int_{\{\mathbf{y}\in\mathcal{R}^d\,:\,|y_1+b|<1\}} \frac{1}{y_1+b} H(\mathbf{y},h)\, d\mathbf{y} \\ &\quad =: I_1(h) + I_2(h),\end{aligned}$$

where

$$H(\mathbf{y},h) := \int_{s=0}^{1} \left(\frac{\partial v}{\partial y_1}(\mathbf{y}) - \frac{\partial v}{\partial y_1}(y_1 - sh, \mathbf{y}^{\perp})\right) ds.$$

It suffices to show that $I_1(h)$ and $I_2(h)$ tend to 0 as h does.

Since f is of controlled decay and we may now assume (8.8) holds, then

$$\lim_{\|\mathbf{y}\|\longrightarrow\infty}\left(\frac{\partial}{\partial y_1}\right)^j v(\mathbf{y})\|\mathbf{y}\|^{d-1+\epsilon}=0$$

for $j=0,1,2$.

Consider $I_1(h)$. In the subregion where $|y_1+b|\geq 1$ and $\|\mathbf{y}\|\geq R$, R suitably large, the integrand $\frac{1}{y_1+b}H(\mathbf{y},h)$ is dominated by $\frac{2\epsilon_0}{|u|(u^2+\rho^2)^{(d-1+\epsilon)/2}}$ where we write $u=y_1+b$ and $\rho=\|\mathbf{y}^{\perp}\|$. Integrating over the subregion, we find that the integral is dominated by

$$\begin{aligned}
&\int_{\{(u,\mathbf{y}^{\perp})\,:\,|u|\geq 1,u^2+\|\mathbf{y}^{\perp}\|^2\geq R^2\}}\frac{2\epsilon_0}{|u|(u^2+\rho^2)^{(d-1+\epsilon)/2}}\rho^{d-2}\,d\rho\,d\Omega\,du\\
&\quad\leq 4\epsilon_0\omega_{d-1}\left(\int_1^{\infty}\frac{du}{u^{1+\epsilon}}\right)\left(\int_0^{\pi/2}\frac{\tan^{d-2}\theta\sec^2\theta d\theta}{\sec^{d-1+\epsilon}\theta}\right)\\
&\quad\leq\frac{4\epsilon_0\omega_{d-1}}{\epsilon}\max\left(\frac{\pi}{2\epsilon},\frac{\pi}{2}\right),
\end{aligned}$$

where the substitution $\rho=u\tan\theta$ has been used as well as the inequality $\cos^{1-\epsilon}\theta\geq(1-\frac{2}{\pi}\theta)^{1-\epsilon}$ when $\epsilon\leq 1$ and $0\leq\theta\leq\frac{\pi}{2}$. Since ϵ_0 can be taken arbitrarily small for R sufficiently large, this term can be made as small as we like. In the remaining subregion, where $|y_1+b|\geq 1$ and $\|\mathbf{y}\|\leq R$, we have

$$\frac{|H(\mathbf{y},h)|}{|y_1+b|}\leq|H(\mathbf{y},h)|$$

and the results of its integration can be made arbitrarily small by taking h sufficiently close to 0 because of uniform continuity of $\frac{\partial v}{\partial y_1}$ on a compact set. So $I_1(h)$ tends to 0 with h.

As for $I_2(h)$, setting

$$G(\mathbf{y},h):=\int_{s=0}^{1}\int_{t=0}^{1}\left(\frac{\partial^2 v}{\partial y_1^2}(-b+t(y_1+b),\mathbf{y}^{\perp})-\frac{\partial^2 v}{\partial y_1^2}(-b-sh+t(y_1+b),\mathbf{y}^{\perp})\right)dt\,ds,$$

we have

$$\begin{aligned}
I_2(h)&=\text{ p.v. }\int_{\{\mathbf{y}:|y_1+b|\leq 1\}}\frac{H(\mathbf{y},h)}{y_1+b}\,d\mathbf{y}\\
&=\int_{\{\mathbf{y}:|y_1+b|\leq 1\}}G(\mathbf{y},h)\,d\mathbf{y}+\text{ p.v. }\int_{\{\mathbf{y}:|y_1+b|\leq 1\}}\frac{H((-b,\mathbf{y}^{\perp}),h)}{y_1+b}\,d\mathbf{y}.
\end{aligned}$$

The principal value integration yields 0 since the order of integration can be changed, and the integral becomes a product of two integrals:

$$\begin{aligned}
&\text{p.v. }\int_{\{\mathbf{y}:|y_1+b|\leq 1\}}\frac{H((-b,\mathbf{y}^{\perp}),h)}{y_1+b}\,d\mathbf{y}\\
&\quad=\left(\int_{\mathcal{R}^{d-1}}H((-b,\mathbf{y}^{\perp}),h)\,d\mathbf{y}^{\perp}\right)\cdot\left(\text{ p.v. }\int_{\{y_1\,:\,|y_1+b|\leq 1\}}\frac{1}{y_1+b}dy_1)\right)\\
&\quad=0,
\end{aligned}$$

where the first factor is well-defined and the second is 0. It remains for us to show that the integral

$$\int_{\{\mathbf{y}:|y_1+b|\leq 1\}}G(\mathbf{y},h)\,d\mathbf{y}$$

tends to zero as h does.

For $\|y\| \geq R$, R suitably large, the integrand is dominated by $\frac{2}{(u^2+\rho^2)^{(d-1+\epsilon)/2}}$ and the integral is dominated by:

$$\begin{aligned}
&\int_{\{(u,\mathbf{y}^\perp)\,:\,|u|\leq 1, u^2+\|\mathbf{y}^\perp\|^2\geq R^2\}} \frac{2}{(\rho^2)^{(d-1+\epsilon)/2}}\rho^{d-2}\,d\rho\,d\Omega\,du \\
&\leq 4\omega_{d-1}\left(\int_{0+}^1 du\right)\left(\int_R^\infty \frac{d\rho}{\rho^{1+\epsilon}}\right) \\
&= \frac{4\omega_{d-1}}{\epsilon R^\epsilon}.
\end{aligned}$$

For R sufficiently large this integral is accordingly negligible. Finally, in the remaining compact subregion $\{\mathbf{y} : |y_1 + b| \leq 1, \|\mathbf{y}\| \leq R\}$ the integrand $G(\mathbf{y}, h)$ can be made as small as we like by choosing h sufficiently close to 0 since $\frac{\partial^2 v}{\partial y_1^2}$ is uniformly continuous there. Accordingly its integral over that region tends to 0 with h, to complete the proof. □

The argument in the odd-dimensional case can be thought of as a variant of the Divergence Theorem [3, p. 423] in d dimensions, in which an integral on the half-space $H^-_{\mathbf{e},b}$ is replaced by an integral on the bounding hyperplane $H_{\mathbf{e},b}$.

Note that the extra condition needed for (8.9) and (8.10) is more stringent than the one in Theorem 4.2 when $|\alpha| \leq d-2$. For (8.3) we can require one less order of differentiability on f than in the Representation Theorem (but at the same time we require that the highest order derivatives vanish to one higher power of $\|\mathbf{x}\|$ at ∞).

Let d be odd. We call a function f *of weakly controlled decay* if $f : \mathcal{R}^d \longrightarrow \mathcal{R}$ is d-times continuously differentiable and $\operatorname{ord} \partial^\alpha f \geq 0$ for all multi-indices α with $0 \leq |\alpha| < d$, and $\operatorname{ord} \partial^\alpha f > d+1$ for all multi-indices with $|\alpha| = d$. Note that neither controlled decay nor weakly controlled decay implies the other.

Proposition 8.2 *Let d be odd. If f is of weakly controlled decay, then f has the representation (4.1) with w_f given by (8.3).*

Proof. Borrowing a technique from [19, p. 1068], we introduce a function ϕ on $\mathcal{R}^d$ that is C^∞ and nonnegative, vanishes for $\|\mathbf{x}\| \geq 1$, and has integral over $\mathcal{R}^d$ equal to 1. Then we define a sequence of functions f_n on $\mathcal{R}^d$ by $f_n(\mathbf{x}) = \int_{\mathcal{R}^d} f(\mathbf{y}) n^d \phi(n(\mathbf{x}-\mathbf{y}))\,d\mathbf{y}$. Each f_n is C^∞, and $\partial^\alpha f_n(\mathbf{x}) = \int_{\mathcal{R}^d} f(\mathbf{y}) \partial^\alpha_{\mathbf{x}}(n^d\phi(n(\mathbf{x}-\mathbf{y}))\,d\mathbf{y} = \int_{\mathcal{R}^d} f(\mathbf{y})(-1)^{|\alpha|}\partial^\alpha_{\mathbf{y}}(n^d\phi(n(\mathbf{x}-\mathbf{y}))\,d\mathbf{y} = \int_{\mathcal{R}^d} \partial^\alpha f(\mathbf{y}) n^d\phi(n(\mathbf{x}-\mathbf{y}))\,d\mathbf{y}$, the last formula holding provided $|\alpha| \leq d$. Since f and all of its derivatives of order $\leq d$ vanish at infinity, it is straightforward to show that f_n converges uniformly to f on $\mathcal{R}^d$ and $\partial^\alpha f_n$ likewise converges uniformly to $\partial^\alpha f$ on $\mathcal{R}^d$ for $|\alpha| \leq d$. If the functions $\{f_n\}$ satisfy the integral formula (4.1) with w_{f_n} as in (8.3), then f will satisfy this integral formula with w_f as in (8.3).

Indeed,

$$\begin{aligned}
f(\mathbf{x}) &= \lim_{n\longrightarrow\infty} f_n(\mathbf{x}) \\
&= \lim_{n\longrightarrow\infty} \int_{S^{d-1}\times\mathcal{R}} w_{f_n}(\mathbf{e},b)\vartheta(\mathbf{e}\cdot\mathbf{x}+b)\,d\mathbf{e}\,db \\
&= \lim_{n\longrightarrow\infty} a_d \int_{S^{d-1}\times\mathcal{R}} \int_{H_{\mathbf{e},b}} D^{(d)}_{\mathbf{e}} f_n(\mathbf{y})\vartheta(\mathbf{e}\cdot\mathbf{x}+b)\,d_H\mathbf{y}\,d\mathbf{e}\,db, \\
&= \int_{S^{d-1}\times\mathcal{R}} w_f(\mathbf{e},b)\vartheta(\mathbf{e}\cdot\mathbf{x}+b)\,d\mathbf{e}\,db
\end{aligned}$$

where Lebesgue's Dominated Convergence Theorem can be applied to move the limit all the way inside. This follows from the fact that

$$\begin{aligned}
\left|D^{(d)}_{\mathbf{e}} f_n(\mathbf{y})\right| &= \left|\int_{\mathcal{R}^d} D^{(d)}_{\mathbf{e}} f(\mathbf{z}) n^d\phi(n(\mathbf{y}-\mathbf{z}))\,d\mathbf{z}\right| \\
&\leq \sup\left\{\frac{C}{(\|\mathbf{z}\|^2+1)^{(d+1+\epsilon)/2}} : \|\mathbf{y}-\mathbf{z}\| \leq 1\right\} \\
&\leq \frac{K}{(\|\mathbf{y}\|^2+1)^{(d+1+\epsilon)/2}}
\end{aligned}$$

for suitable constants C and K that are independent of n and $\mathbf{e}$. Since for $d \geq 3$

$$\begin{aligned}
&\int_{S^{d-1}\times\mathcal{R}} \int_{H_{\mathbf{e},b}} \frac{1}{(\|\mathbf{y}\|^2+1)^{(d+1+\epsilon)/2}} \vartheta(\mathbf{e}\cdot\mathbf{x}+b)\, d_H\mathbf{y}\, d\mathbf{e}\, db \\
&= \omega_{d-1} \int_{S^{d-1}\times\mathcal{R}} \int_{[0,\infty)} \frac{\rho^{d-2}\, d\rho \vartheta(\mathbf{e}\cdot\mathbf{x}+b)\, d\mathbf{e}\, db}{(b^2+\rho^2+1)^{(d+1+\epsilon)/2}} \\
&\leq \omega_{d-1} \frac{\pi}{2} \int_{S^{d-1}\times\mathcal{R}} \frac{\vartheta(\mathbf{e}\cdot\mathbf{x}+b)\, d\mathbf{e}\, db}{(b^2+1)^{(2+\epsilon)/2}} \\
&\leq \omega_{d-1}\omega_d \frac{\pi}{2} \int_{-\|\mathbf{x}\|}^{\infty} \frac{db}{(b^2+1)^{(2+\epsilon)/2}} \\
&< \infty,
\end{aligned}$$

the dominating function is integrable. The case $d = 1$ can be worked out trivially, but this exercise is superseded by Proposition 5.1.

We must still establish that $\{f_n\}$ satisfies the hypotheses of the Representation Theorem. By Lemma 4.1 it suffices to show that the order $|\alpha| = d+1$ derivatives of f_n satisfy ord $\partial^\alpha f_n > d+1$. However, for $|\alpha| = d+1$, using the extra increment in the order of vanishing, we have:

$$\begin{aligned}
|\partial^\alpha f_n(\mathbf{x})| &= \left| \int_{\mathcal{R}^d} f(\mathbf{y}) \partial^\alpha(\phi(n(\mathbf{x}-\mathbf{y}))n^d\, d\mathbf{y} \right| \\
&= \left| \int_{\mathcal{R}^d} \partial^\beta f(\mathbf{y}) \frac{\partial\phi}{\partial u_i}(n(\mathbf{x}-\mathbf{y}))n^{d+1}\, d\mathbf{y} \right| \\
&\leq C_n \sup\left\{ \frac{1}{(\|\mathbf{y}\|^2+1)^{(d+1+\epsilon)/2}} : \|\mathbf{x}-\mathbf{y}\| \leq 1 \right\} \\
&\leq \frac{D_n}{(\|\mathbf{x}\|^2+1)^{(d+1+\epsilon)/2}}
\end{aligned}$$

where $\alpha = \beta + \mathbf{u}_i$, $\mathbf{u}_i$ a coordinate vector with 1 in the i-th position and 0 elsewhere for some i such that $\alpha_i \geq 1$. □

Proposition 8.2 generalizes the result of Kůrková, Kainen and Kreinovich [19]. Proposition 8.1, (8.3) and (8.7), extends the result of Ito [13], while Proposition 8.1, (8.9) and (8.10), extends the result of Carroll and Dickinson [4].

9 Discussion

The history of the representations above is of some interest. Helgason's book [11] offers generalizations and at the same time points back to antecedent ideas, including papers of Funk and Radon. Gel'fand, Graev, and Vilenkin [9] also obtained a Radon-type representation. Indeed the history of this representation probably extends back beyond Radon and Hilbert to such figures as Cauchy, Poisson, and Laplace.

Properties of the weight function in the integral formula can be developed further. In our present setting w_f is a continuous function on $S^{d-1} \times \mathcal{R}$, is integrable on this set (along with $(\mathbf{e}, b) \longmapsto w_f(\mathbf{e}, b)\theta(\mathbf{e}\cdot\mathbf{x}+b)$) and satisfies $\lim_{|b|\longrightarrow\infty} w_f(\mathbf{e}, b)|b|^{1+\epsilon} = 0$. In the proof of Theorem 4.2 we found a class of weight functions $(\mathbf{e}, b) \longmapsto \hat{w}_f(\mathbf{e}, b) + \hat{w}_f(-\mathbf{e}, -b)$, each of which represents the zero function. Since w_f is not unique, one can seek choices for it that minimize various measures of cost.

It is apparent that the representation applies to functions other than those given in Theorem 4.2, and Propositions 8.1 and 8.2. Ito [13] points out that the conditions on the functions to be approximated can be loosened considerably but does not provide details. In the one-dimensional case, Proposition 5.1 and Lemma 6.1 both demonstrate that the growth conditions can be weakened, or even abandoned. In the proof of the Representation Theorem use was made of representations of $\|\mathbf{x}\|$ and $\beta(\|\mathbf{x}\|)$ by integral combinations of Heavisides. Similar representations can be made for any polynomial $\mathbf{x} \longmapsto p(\|\mathbf{x}-\mathbf{y}\|)$.

Also of interest is how a finite sum approximating the integral formulas can be selected (choices of weights and half-spaces). “A *quadrature formula* is a numerical rule whereby the value of a definite integral is approximated by the use of information about the integrand only at discrete points (where the integrand is defined)” (Engels, [7, p. 1]). A quadrature of the integral formula from Theorem 4.2 would determine parameters of a Heaviside perceptron network that should be useful information for designing a learning algorithm. Elsewhere we have shown that for every $n \geq 1$, integrable functions f on $[0,1]^d$ have best approximations by combinations of n or fewer Heavisides ([17], [14]), but these best approximations cannot vary continuously with f ([18], [15]).

Perhaps quadrature can be achieved by an algorithm which first chooses among distinct alternatives and then proceeds continuously.

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