Continuous First-Order Constraint Satisfaction with Equality and Disequality Constraints

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Abstract. In an earlier paper we have shown, how one can successfully use constraint satisfaction techniques for proving and solving formulae in the first-order predicate language over the real numbers (i.e., real first-order constraints). This approach was restricted to inputs that contain inequality symbols such as $\leq$, but no equality symbols ($=$) or disequality symbols ($\neq$). In this paper we lay the basis for extending this approach to inputs that contain (dis)equalities. This considerably widens the practical applicability of numerical constraint satisfaction methods.

1 Introduction

Let a real first-order constraint be a formula in the first-order predicate language with predicate symbols $=$, $\neq$, $\leq$, $<$, function symbols such as $+$, $\times$, sin, exp, and their usual interpretation over the real numbers. The problem of solving real first-order constraints is of fundamental importance—we have created a web-page [4] that lists more then 50 papers with applications. In an earlier paper [5] we have shown how one can extend constraint satisfaction techniques to solve real first-order constraints that do not contain equalities or disequalities—removing some of the problems of the classical quantifier elimination algorithms [2]. In this paper we set the basis for extending this approach to the case with equalities and disequalities.

The main difficulty for introducing equalities in this context is, that the solution set of equality constraints does not have volume, which can make it very difficult to find elements in such a solution set. However, it is often quite easy to find a small interval that provably contains a solution. For example, the expression $x^2 - 2$ is negative for $x = 0$ and positive for $x = 4$. Hence, by elementary analysis (Boltzmann intermediate value theorem), the interval $[0, 4]$ contains a solution of $x^2 - 2 = 0$, and $\exists x\, x^2 - 2 = 0$ is true. Note that this approach fails if we use the larger interval $[-4, 4]$, where the function is positive also for $x = -4$.

By applying first-order constraint satisfaction [5, 1] one can find intervals that enclose the solutions of the occurring equalities tightly. But, up to now it was unclear, how to extend the approach sketched above to general first-order constraint for which all the variables ranges over small closed intervals.

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2 Main Idea

In this section we demonstrate the main idea of how to (dis)prove first-order constraints (short: constraints) with small quantifier bounds. For this we employ a semi-formal style. In the following section we will then describe the formal details.

We fix a totally ordered set $\mathcal{V} = \{x_1, \dots, x_{|\mathcal{V}|}\}$ of variables. For any constraint ϕ and constants $a_1, \dots, a_{|\mathcal{V}|}$, we denote by $\phi(a_1, \dots, a_{|\mathcal{V}|})$ the result of substituting $a_1, \dots, a_{|\mathcal{V}|}$ into ϕ (note that this only changes the free variables). For an interval I, we denote by $\underline{I}$ its lower bound, and by $\overline{I}$ its upper bound. Given a box $B \subseteq \mathbb{R}^{|\mathcal{V}|}$ and a variable $u \in \mathcal{V}$, we denote by $B(u)$ the coordinate of B corresponding to u according to the order on $\mathcal{V}$.

We proceed recursively according to the structure of constraints. For example, for a constraint of the form $\exists x \in I_x\ \forall y \in I_y\ \exists z \in I_z\ x^2yz = 0\ \wedge\ xy^2z - 1 = 0$, we first compute some information for the atomic sub-constraints $x^2yz = 0$ and $xy^2z - 1 = 0$, then for $x^2yz = 0 \wedge xy^2z - 1 = 0$, $\exists z \in I_z\ x^2yz = 0 \wedge xy^2z - 1 = 0$, and so on. Along the way we try prove existential quantifiers or disprove universal quantifiers using this information.

Which information allows us to do this? Let us first consider a constraint of the form $\exists x \in I\ \phi$. In this case, knowing that ϕ has a solution in I suffices to prove the constraint However, information about one solution for all atomic sub-constraints is not enough: Consider a constraint of the form $\exists x \in I_x \exists y \in I_y\ \phi_1 \wedge \phi_2$. From knowing that both ϕ_1 and ϕ_2 have a solution within $I_x \times I_y$, we cannot infer that $\phi_1 \wedge \phi_2$ has a solution in $I_x \times I_y$, because the respective solutions might be different.

Thus we have to propagate more information for proving existentially quantified conjunctions. In the example above, from knowing that *for all y in I_y there is an x in I_x such that ϕ_1* and that *for all x in I_x there is a y in I_y such that ϕ_2*, and furthermore knowing that the solution for the existentially quantified variable depends continuously on the universally quantified one, we can prove $\exists x \in I_x \exists y \in I_y\ \phi_1 \wedge \phi_2$ (in other words, on the left-hand side of Figure 1, there is a line going from left to right, and a line going from top to bottom, and therefore there is a point on both lines).

One might think that information similar to the above is enough. But while it would be sufficient in the example above, in general, it does not propagate. For illustrating this, we assume that in the above example both ϕ_1 and ϕ_2 contain an additional parameter p, and assume that we have the information *for all p in I_p for all y in I_y there is an x in I_x such that ϕ_1* and *for all p in I_p for all x in I_x there is a y in I_y such that ϕ_2*. This information allows us to prove $\forall p \in I_p \exists x \in I_x \exists y \in I_y\ \phi_1 \wedge \phi_2$, but it does *not* allow us to propagate the according continuity information: In general, the x- and y-coordinates of solutions of $\phi_1 \wedge \phi_2$ might not depend continuously on p! For example, the p- and x-coordinates of the solution might look like the right-hand side of Figure 1, where for all p we can find an according x in the solution, but this solution can not be described by a continuous function.

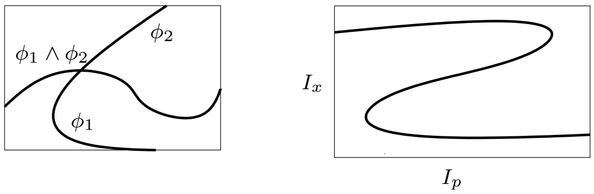

Fig. 1. Information Propagation

In order to deal with this problem we use even more general information to describe such solutions: Given a constraint ϕ, a variable set $\{u_1, \ldots, u_k\} \subseteq \mathcal{V}$ and a box $B \subseteq \mathbb{R}^{|\mathcal{V}|}$, the information that we propagate is the predicate "there exist continuous $F_{x_1}, \ldots, F_{x_{|\mathcal{V}|}}$ (*witness functions*) in $\mathbb{R}^k \to \mathbb{R}$ such that for all $(s_1, \ldots, s_k) \in [0, 1]^k$,

- for all $i \in \{1, \ldots, k\}$, $s_i = 0$ implies $F_{u_i}(s_1, \ldots, s_k) = \underline{B(u_i)}$,
- for all $i \in \{1, \ldots, k\}$, $s_i = 1$ implies $F_{u_i}(s_1, \ldots, s_k) = \overline{B(u_i)}$,
- for all $i \in \{1, \ldots, |\mathcal{V}|\}$, $F_{x_i}(s_1, \ldots, s_k) \in B(x_i)$,
- $\phi(F_{x_1}(s_1, \ldots, s_k), \ldots, F_{x_{|\mathcal{V}|}}(s_1, \ldots, s_k))$ is true."

The first item guarantees that there is a witness on the lower bound (in the right-hand side of Figure 1, the solution touches the box on the left-hand side). The second item does the same for the upper bound (the solution touches the box on the right-hand side). The third item requires the witness function to stay within the box, and the fourth item requires the witness function to describe a solution.

In the following we denote the above predicate by $M_\forall(\{u_1, \ldots, u_k\}, \phi, B)$. We call such a predicate *witness predicate*, and k its *dimension*. This predicate implies the first-order predicate $\forall u_1 \in B(u_1) \ldots \forall u_k \in B(u_k) \exists v_1 \in B(v_1) \ldots \exists v_l \in B(v_l)\ \phi$, where $\{v_1, \ldots, v_l\} = \mathcal{V} \setminus \{u_1, \ldots, u_k\}$. Moreover, for a closed constraint ϕ, a set of variables U and box B, $M_\forall(U, \phi, B)$ implies that ϕ holds. So this is the information we want to compute for proving the total constraint.

For disproving universal quantifiers, we take the negation of the input constraint, and apply the above procedure to the resulting existential quantifiers.

Note that the above approach will fail if the involved intervals are too large: First, the solution of the constraint might behave too wildly, making it impossible to describe it by witness functions (e.g., for the constraint $x^2 + y^2 = 1$ within $[-2, 2] \times [-2, 2]$ no witness predicate with dimension higher than 0 holds). And second, the larger a box, the higher is the effort to find such witness functions

3 Computing and Propagating Witness Predicates

In this section we show formally how one can propagate witness predicates over the logical symbols $\wedge$, $\vee$, $\exists$, and $\forall$.

Theorem 1.

– *If $M_\forall(U_1, \phi_1, B)$, $M_\forall(U_2, \phi_2, B)$, and $U_1 \cup U_2 = \mathcal{V}$, then $M_\forall(U_1 \cap U_2, \phi_1 \wedge \phi_2, B)$.*
– *If $M_\forall(U_1, \phi_1, B)$ and $M_\forall(U_2, \phi_2, B)$ then $M_\forall(U_1, \phi_1 \vee \phi_2, B)$ and $M_\forall(U_2, \phi_1 \vee \phi_2, B)$.*
– *If $M_\forall(U, \phi, B)$ then $M_\forall(U \setminus \{x\}, \exists x \in B(x)\ \phi, B)$.*
– *If $M_\forall(U, \phi, B)$ where U contains all the free variables of ϕ then $M_\forall(U, \forall x \in B(x)\ \phi, B)$.*

Proof. We just show the first case, the rest is easy. The idea of the proof is that whenever the witness functions of ϕ_1 and of ϕ_2 have the same value, then this is a solution for $\phi_1 \wedge \phi_2$, and so the fourth property necessary for witness functions is fulfilled. We will prove that there exist such values and construct witness functions from them that fulfill the remaining properties.

We assume that in the order on $\mathcal{V}$ the elements of $U_1 \cap U_2$ appear first. Let $F^1_{x_1}, \ldots, F^1_{x_{|\mathcal{V}|}}$ be the witness functions of ϕ_1, and let $F^2_{x_1}, \ldots, F^2_{x_{|\mathcal{V}|}}$ be the witness functions of ϕ_2. For all $i \in \{1, \ldots, |\mathcal{V}|\}$, let $\mathcal{F}^1_{x_i} : [0,1]^{|U_1|+|U_2|} \to \mathbb{R}$ be such that $\mathcal{F}^1_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) := F^1_{x_i}(r_1, \ldots, r_{|U_1|})$ and let $\mathcal{F}^2_{x_i} : [0,1]^{|U_1|+|U_2|} \to \mathbb{R}$ be such that $\mathcal{F}^2_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) := F^2_{x_i}(r_{|U_1|+1}, \ldots, r_{|U_1|+|U_2|})$.

Let $\mathcal{G} : [0,1]^{|U_1|+|U_2|} \to \mathbb{R}^{|\mathcal{V}|}$ be the function for which the i-th component is $\mathcal{F}^1_{x_i} - \mathcal{F}^2_{x_i}$, where $i \in \{1, \ldots, |\mathcal{V}|\}$. For all elements $(r_1, \ldots, r_{|U_1|+|U_2|})$ of the solution set of $\mathcal{G} = 0$, $\phi_1(\mathcal{F}^1_{x_1}(r_1, \ldots, r_{|U_1|+|U_2|}), \ldots \mathcal{F}^1_{x_{|\mathcal{V}|}}(r_1, \ldots, r_{|U_1|+|U_2|}))$ and $\phi_2(\mathcal{F}^1_{x_1}(r_1, \ldots, r_{|U_1|+|U_2|}), \ldots, \mathcal{F}^1_{x_{|\mathcal{V}|}}(r_1, \ldots, r_{|U_1|+|U_2|}))$ hold. So for these values $\phi_1 \wedge \phi_2$ holds, too. Now we show that such solutions of $\mathcal{G} = 0$ exist, and that we can describe them by witness functions.

Observe that the dimension of the co-domain of $\mathcal{G}$ is $|\mathcal{V}| = |U_1 \cup U_2| = |U_1| + |U_2| - |U_1 \cap U_2|$, which allows us to interpret the first $|U_1 \cap U_2|$ variables of $\mathcal{G}$ as parameters. Now we apply a parametric version of Miranda's theorem (Theorem 5.3.7 of Neumaier [3]). It suffices to prove that there is a bijection C between the variables $x_1, \ldots, x_{|\mathcal{V}|}$ and the coordinates of $\mathcal{G}$ that are no parameters, such that for an arbitrary, but fixed $i \in \{1, \ldots, |\mathcal{V}|\}$, for all $(r_1, \ldots, r_{|U_1|+|U_2|}) \in [0,1]^{|U_1|+|U_2|}$,

$$r_{C(i)} = 0 \text{ implies } \mathcal{F}^1_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) - \mathcal{F}^2_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) \leq 0, \text{ and} \quad (1)$$

$$r_{C(i)} = 1 \text{ implies } \mathcal{F}^1_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) - \mathcal{F}^2_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) \geq 0. \quad (2)$$

Here we have two cases:

– $x_i \in U_2$: In this case, let $C(i) := |U_1| + k$, where k is the number of x_i in U_2 according to the order on $\mathcal{V}$. Therefore $C(i)$ points into the witness functions of ϕ_2, and according to the properties of these witness functions $r_{C(i)} = 0$ implies $\mathcal{F}^2_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) = \underline{B(i)}$, and $r_{C(i)} = 1$ implies $\mathcal{F}^2_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) = \overline{B(i)}$. As a consequence, since $\mathcal{F}^1_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) \in B(i)$, the Inequalities 1 and 2 hold.
– Otherwise ($x_i \in U_1$ but not in U_2): In this case let $C(i)$ be the number of x_i in U_1. Since all variables in $U_1 \cap U_2$ appear before in the order, $C(i) > |U_1 \cap U_2|$, and it points to the coordinates of $\mathcal{G}$ that are no parameters.

So according to the properties of these witness functions $r_{C(i)} = 0$ implies $\mathcal{F}^1_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) = \underline{B(i)}$, and $r_{C(i)} = 1$ implies $\mathcal{F}^1_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) = \overline{B(i)}$. As a consequence, since $\mathcal{F}^2_{x_i}(r_1, \ldots, r_{|U_1|+|U_2|}) \in B(i)$, the Inequalities 1 and 2 hold (up to a insignificant change of signs).

Now by the parametric Miranda's theorem there is a continuous function $S : [0,1]^{|U_1 \cap U_2|} \to [0,1]^{|\mathcal{V}|}$ such that for all $(p_1, \ldots, p_{|U_1 \cap U_2|}) \in [0,1]^{|U_1 \cap U_2|}$, $\mathcal{G}(p_1, \ldots, p_{|U_1 \cap U_2|}, S(p_1, \ldots, p_{|U_1 \cap U_2|})) = 0$. Therefore, the functions $\lambda p_1, \ldots, p_{|U_1 \cap U_2|} \cdot \mathcal{F}^1_{x_i}(p_1, \ldots, p_{|U_1 \cap U_2|}, S(p_1, \ldots, p_{|U_1 \cap U_2|}))$, where $i \in \{1, \ldots, |\mathcal{V}|\}$, provide the witness functions for $M_\forall(U_1 \cap U_2, \phi_1 \wedge \phi_2, B)$. It is easy to show that they fulfill the necessary properties. □

It is easy to check that the above theorem is optimal in the sense that in general $M_\forall$ does not hold for supersets of the ones provided.

Given a first-order constraint with small quantification bounds we can now compute witness predicates for atomic constraints, and use Theorem 1 to propagate this information to witness predicates for the total constraint. Any witness predicate for then total constraint proves it. For disproving a constraint we do the same on the negation of the constraint. Whenever we succeed to compute a witness predicate for the negation, we have disproven the original constraint.

4 Conclusion

In this paper we have lain the basis for using constraint programming techniques for solving first-order constraints that contain (dis)equalities. For building a successful solver it remains to combine the result with continuous first-order constraint satisfaction [5], which can compute a constraint that is equivalent to the input, but for which the variables range over small intervals. The main problem is, to determine how small the intervals of which variables should be, such that one can efficiently compute the necessary witness functions.

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